ADVANCES IN MATHEMATICS 39, 289–329 (1981)

Integral Ring Homomorphisms

M. ARTIN AND W. SCHELTER*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

The purpose of this paper is to extend the classical notion of integral extensions of commutative rings to homomorphisms of affine pi rings. It generalizes previous work [R. Paré and W. Schelter, *J. Algebra* 53 (1978), 477–479; W. Schelter, J. Algebra 40 (1976), 245–254; errata 44 (1979), 576], which assumed the existence of centralizing elements. Without such elements the close relation between integrality and module-finiteness breaks down (cf. Examples 1.3, 1.4), but the geometric implications of integrality remain. Our main result, Theorem 6.3, is an analogue of Chevalley's theorem that a homomorphism $R \to S$ is integral if and only if the induced map on spectra is proper. It is proved in Sections 6–8 by combining a geometric analysis with some explicit estimates of degrees. A corollary is that the composition of integral homomorphisms is integral (Proposition 6.10). The main new tools we use are central homomorphisms of a ring R to orders over Dedekind domains, which we call *curves*. We prove (Theorem 5.13) that the boundary of any constructible set in $X = \operatorname{Spec} R$ contains a dense set of points *accessible* along such a curve. This fact contains a description (Corollary 5.14) of the Zariski topology on X, as well as the theorem of Bergman and Small on degrees of representations (Corollary 5.15). Some of the results of this paper were announced in [M. Artin *in* "Proceedings, International Symposium on Algebraic Geometry, Kyoto, 1977," pp. 237–247].

Contents.

1. INTEGRALITY

Throughout this paper, the word *ring* will mean an affine algebra over an algebraically closed field k, i.e., a finitely generated k-algebra satisfying the identities of $n \times n$ matrices for some n. By *spectrum*, Spec R of a ring R, we mean the space of maximal ideals of R, with its Zariski topology.

We use square brackets to denote adjunction of central elements to a ring. For example, if u, v are variables, then $k[u, v]$ denotes the commutative polynomial ring in u, v. Curly brackets are used when no assumption of

* Supported in part by the National Science Foundation and the National Research Council of Canada.

0001-8708/81/030289-41$05.00/0

commutativity is made. Thus if R is a subring of a ring S and $\alpha \in S$, then $R\{\alpha\}$ denotes the subring of S generated by R and α.

A ring homomorphism $R \to^{\varphi} S$ is called *integral*, if every element of S satisfies a monic polynomial equation with coefficients in R. More precisely, φ is integral if for each $s \in S$, there is a polynomial $p(x)$ in the free polynomial ring in one variable which is *monic*, i.e., has a power of x as its highest degree term, and such that $p(s) = 0$. The phrase *free polynomial ring* means that we use the word polynomial in a generalized sense, in which coefficients from R may be interspersed among the variables. An example of a monic polynomial equation is

$$x^3 + axbx + cx + xd + e = 0.$$

Obviously this definition reduces to the classical one when R, S are commutative. In particular, a commutative ring S will be integral if it is a finite R-module. It is much less obvious that this is true for an arbitrary *extension*, i.e., a ring homomorphism $R \to^{\varphi} S$ such that S is generated over R by its centralizer

$$S^R = \{s \in S \mid rs = sr, \text{ all } r \in R\}.$$

THEOREM 1.1. *An extension S of R is integral over R if and only if S is a finite (left or right) R-module.*

This theorem is the main result of [8]. In that paper it is shown that for any ring R, not necessarily a pi ring, the ring $S = M_n(R)$ of $n \times n$ matrices with entries in R is integral over R. The "if" part of the theorem follows easily from that fact, and the converse from Shirshov's theorem. (Unfortunately the standard references such as [9, p. 152] for Shirshov's theorem assume commutativity of R. However the proof is easily generalized.)

Integral extensions were studied in [8, 12]. In this paper we consider integrality for arbitrary homomorphisms of affine pi rings.

PROPOSITION 1.2. *Suppose that R is left noetherian and that S is a finite left R-module. Then S is integral over R.*

Proof. Let $s \in S$. The R-submodule of S generated by $\{1, s, s^2, \ldots\}$ is finitely generated, say by $\{1, \ldots, s^l\}$. Then $s^{l+1} = \sum_0^l r_i s^i$. Therefore s is integral over R.

The next two examples show that in general, being a finitely generated R-module need not imply, nor be implied by, integrality over R (see also [2, p. 244]).

EXAMPLE 1.3. A finite module which is not integral. Let $A = k[u]$ be a polynomial ring over a field in one central variable u, and let $B = k[u, u^{-1}]$. Let

$$R = \begin{bmatrix} A & 0 \\ B & B \end{bmatrix}, \qquad S = \begin{bmatrix} B & B \\ B & B \end{bmatrix}.$$

Then

$$S = e_{22}R + e_{12}R,$$

but S is not integral over R. In fact, the matrix $u^{-1}I$ is not integral over R, so the subring $R[u^{-1}]$ of S is not integral. In this example, the map $\operatorname{Spec} R[u^{-1}] \to \operatorname{Spec} R$ is not a closed map, nor surjective. We will show (Proposition 7.1, Corollary 7.2) that both properties hold for injective integral homomorphisms.

EXAMPLE 1.4. An integral homomorphism in which S is generated over R by a single element, but is not a finite R-module. Let $A = k[u, v]$ be a polynomial ring in two central variables over a field k. Let

$$R = \begin{pmatrix} A & uA \\ uA & k + u^2A \end{pmatrix}, \qquad S = \begin{pmatrix} A & A \\ uA & k + uA \end{pmatrix}.$$

Then S is integral over R. For if $s \in S$, we may choose $r \in R$ so that

$$s - r \in \begin{pmatrix} 0 & A \\ 0 & uA \end{pmatrix}.$$

Then $(s - r)^2 = r' \in R$, and s satisfies the monic equation

$$s^2 - sr - rs + (r^2 - r') = 0.$$

Also, note that S is generated over R by the matrix unit e_{12}, that is, $S = R\{e_{12}\}$, and that $R = k\{e_{11}, ue_{12}, ue_{21}\}$. But to generate S as left R-module, the infinite set $\{uv^n e_{22}\}$ is required. The spectra of the two rings in this example are homeomorphic.

In Section 7 we characterize integrality of a homomorphism $R \to S$ in terms of the induced correspondence $\operatorname{Spec} S \to \operatorname{Spec} R$ (see Section 4). Namely, the correspondence must be closed (Proposition 7.1), and the same must be true for the homomorphism $R[v] \to S[v]$ obtained by adjoining a central variable v. One dimensional rings provide good examples for visualizing the geometry of the correspondence.

EXAMPLE 1.5. Let $R = k[x]$, $S = M_2(k[u])$, and consider the map $R \to^{\varphi} S$ defined by

$$x \rightsquigarrow \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k[u]).$$

Both spectra are affine lines: $\operatorname{Spec} R$ is the x-line and $\operatorname{Spec} S$ is homeomorphic to the u-line. The graph of the correspondence $\operatorname{Spec} S \to \operatorname{Spec} R$ induced by φ (see Section 4) is the locus in the (x, u)-plane defined by the Cayley–Hamilton equation

$$x^2 - (\operatorname{tr} \alpha)\, x + (\det \alpha) = 0.$$

The ring S has Krull dimension 1. To determine whether or not φ is integral, Theorem 6.3 shows that it is enough to check that its center $Z(S) = k[u]$ is integral over R, i.e., that u is integral. The above equation shows that this will be true if and only if $\det \alpha$ has higher degree in u than $\operatorname{tr} \alpha$, which is usually the case. Thus, if φ is defined by

$$\alpha = \begin{pmatrix} u & 0 \\ 0 & u+1 \end{pmatrix} \tag{1.6}$$

then S is integral over R, while if

$$\alpha = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \tag{1.7}$$

it is not integral. The graph of the correspondence in case (1.7) is the union of the two lines $\{x = 0\}$, $\{x = u\}$. It happens to be a closed correspondence, but ceases to be so when an auxiliary central variable v is adjoined. For instance, the closed set $\{uv = 1\}$ in the (u, v)-plane has image $\{x = 0, v \neq 0\}$ in the (x, v)-plane under this correspondence.

For non-extensions, as Examples 1.3 and 1.4 show, integrality is not the same as being a finitely generated module. Thus the usual method of verifying transitivity of integrality does not work. However it is true that the composition of closed maps is closed, and using this we are able to show that the composition of integral maps is integral (Theorem 6.3, Proposition 6.10). As Small has remarked to us, an extension of this result to arbitrary rings would solve well-known open problems. For instance, a matrix algebra over an algebraic k-algebra is a composition of integral extensions, but is not known to be algebraic unless k is uncountable.[1]

As another possible definition for integral one might demand this: Let $\{s_1, \ldots, s_n\}$ be a finite subset of S. There is an integer N such that any monomial $r_0 s_{i_1} r_i s_{i_2} \cdots s_{i_m} r_m$, with $r_i \in R$, can be expressed as a sum of

[1] *Note added in proof.* G. Bergman has recently found an example in which transitivity of integrality breaks down.

monomials of degree less than N in the s_i's. Certainly this condition implies integrality of each element $s \in S$. Our main theorem (Theorem 8.1) tells us that it is equivalent to integrality.

It is important to realize that, in order to check that $R\{\alpha\}$ is integral over R, one must do more than just check that α is integral. For example, let $R = M_2(k[u])$ and let $\alpha = u^{-1}e_{12}$. Then $\alpha^2 = 0$, so α is integral, but $R\{\alpha\} = M_2(k[u, u^{-1}])$, and $u^{-1} \in R\{\alpha\}$ is not integral over R. Suppose however that α is central. Then if α is integral, $R\{\alpha\}$ is a finite R-module and hence is integral (Theorem 1.1). More generally if $R\{\alpha_1, \ldots, \alpha_m\}$ is an extension of pi degree n, and the α_i centralize R, then checking integrality can be done by verifying integrality of monomials of degree less than or equal to n^2 in the α_i (Shirshov's theorem). The following example illustrates that a direct extension of Shirshov's theorem to the general case is not possible (see however Theorem 9.3).

EXAMPLE 1.8. A ring S which is not integral over a subring R, but such that small monomials in the generators are integral. Let $S = M_2(k[u, u^{-1}])$ and let R be the subring generated by $\alpha_1 = ue_{11}$, $\bar{\alpha}_1 = u^{-1}e_{11}$, $\alpha_2 = ue_{22}$. Then for any integer $k \geqslant 0$, S is generated over R by $\beta = u^k(e_{12} + e_{21})$, because $\bar{\alpha}_1^k\beta = e_{12}$, $\beta\bar{\alpha}_1^k = e_{21}$, etc. The element $\bar{\alpha}_2 = u^{-1}e_{22}$ is obviously not integral over R, but any monomial in $\{\alpha_1, \bar{\alpha}_1, \alpha_2, \beta\}$ of degree $\leqslant k$ is integral.

We omit the proof of the following proposition.

PROPOSITION 1.9. (i) *Let $R \to^{\varphi} S$ be an integral homomorphism and let T be a subring of S containing $\varphi(R)$. Then $R \to T$ is integral.*

(ii) *Consider a commutative diagram of ring homomorphisms:*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ \bar{R} & \xrightarrow{\bar{\varphi}} & \bar{S}. \end{array}$$

If the vertical arrows are surjective and φ is integral, then $\bar{\varphi}$ is integral.

(iii) *Let $R \to^{\varphi} S$ be a ring homomorphism and let $N \subset S$ be a nilideal. If $R \to S/N$ is integral, so is $R \to S$.*

PROPOSITION 1.10. *If S is an integral extension of a subring R, then the map* Spec $S \to$ Spec R *is surjective and* $\dim R = \dim S$. *Prime ideals of R have prime ideals of S lying over them.*

This is proved in [12, Theorem 1]. It can also be deduced from Theorem 1.1 and the Nakayama lemma [1, (5.2)]: By Theorem 1.1, S is a finite central R-bimodule. Since $R \subset S$, the annihilator of S as an R-module is zero. If $\mathfrak{p} \subset R$ is any prime ideal, then the annihilator of $\bar{S} = S/\mathfrak{p}S$ as $\bar{R} = R/\mathfrak{p}$-module is zero, by Nakayama. Hence $\bar{R} \subset \bar{S}$. The standard argument now shows that there is a prime ideal $\bar{\mathfrak{q}}$ in $\bar{S}$ with $\bar{\mathfrak{q}} \cap \bar{R} = (0)$; hence a prime $\mathfrak{q} \subset S$ with $\mathfrak{q} \cap R = \mathfrak{p}$ (see, for example, [1, (7.4)]). The assertion on dimension follows easily from this.

2. Orders

This section is a review of well-known facts, and we omit some of the proofs. Throughout the section, the symbol D will denote a finitely generated Dedekind domain, and K will denote its field of fractions. The spectrum Spec D is a smooth affine algebraic curve [5]. We want to have analogues of curves in the context of pi rings, and natural ones are furnished by *D-orders* A in the algebra $M_n(K)$ of $n \times n$ matrices over K. These are the subrings A of $M_n(K)$, which are finite D-modules such that $K \otimes_D A \approx M_n(K)$. In this paper, the word order is used only for such orders in matrix algebras.

Dedekind domains are the integrally closed commutative domains of dimension 1. Since by Tsen's theorem [15] every central simple K-algebra is isomorphic to $M_n(K)$ for some n, orders occupy an analogous position among pi rings.

PROPOSITION 2.1. *Let R be a semi-prime affine ring of Krull dimension* 1. *Then*

(i) *R is a finite module over its center Z, and Z is a finitely generated, semi-prime ring of dimension* 1.

(ii) *R is an order if and only its center Z is a Dedekind domain.*

If one wants to allow arbitrary ground fields, the definition of order should be extended to include subrings of $M_n(L)$, where L is a division ring finite over its center K. Only minor modifications are required at other points in this paper.

Rings of dimension 1 also have these properties:

PROPOSITION 2.2. (i) *Let A be a D-order in $M_n(K)$, and let A' be any finitely generated subring of $M_n(K)$ containing A. Then A' is an order over its center D'.*

(ii) *Let R be a prime affine ring of dimension* 1. *Every prime k-algebra $R' \subset R$ having the same pi degree is affine. The set of such subrings satisfies the ascending chain condition.*

We will sketch the proofs of these propositions because we do not know of a reference which treats them all together.

LEMMA 2.3. *Let D be a finitely generated Dedekind domain, and let $C = \operatorname{Spec} D$. The finitely generated D-subalgebras D' of K are Dedekind domains. There is a* 1–1 *correspondence of these rings with finite subsets $S \subset C$ having the property that* $\operatorname{Spec} D' \approx C - S$. *A D-subalgebra D' of K is finitely generated if and only if its spectrum has that form.*

Proof. Any torsion-free D-module M is the intersection of its localizations: $M = \bigcap M_p$, where p runs over points of C. Thus the set $\{D'_p\}$ of localizations determines the subring D'. Since D_p is a discrete valuation ring, there is no ring between D_p and K, and so D'_p is either D_p or K. From this one deduces easily that $\operatorname{Spec} D' \approx \{p \in C \mid D'_p = D_p\}$, hence that $C' = \operatorname{Spec} D'$ determines D', and moreover that D' is integrally closed. If D' is finitely generated, one can choose a common denominator c for the generators: hence $D' \subset D[c^{-1}]$. Therefore C' contains $\operatorname{Spec} D[c^{-1}]$, which, since the dimension is 1, is the complement of a finite set of C. So C' is the complement of a finite set too. Conversely, given any finite set $S \subset C$, there is an element $a \in K$ which has poles at every point of S but nowhere else. Then $\operatorname{Spec} D[a] = C - S$.

LEMMA 2.4. *Let R be an affine, semi-prime, commutative ring of dimension* 1. *Then every k-algebra $R' \subset R$ is affine.*

Proof. Assume that R is prime. The field of fractions K' of R' is finitely generated, so we may choose a finitely generated subring R_0 of R' with field of fractions K'. Let D_0, D be the integral closures of R_0, R, respectively, and let $D' = R'D_0$:

$$\begin{array}{ccccc} R_0 & \subset & R' & \subset & R \\ \cup & & \cup & & \cup \\ D_0 & \subset & D' & \subset & D. \end{array}$$

It is not assumed that $K' = \operatorname{Fract} D$. Nevertheless, $\operatorname{Spec} D$ covers an open set of $\operatorname{Spec} D_0$ (i.e., all but a finite set) because D is finitely generated. Therefore D' is a finitely generated Dedekind domain, by the previous lemma. That being so, we can adjoin finitely many elements to R_0, to arrive at a situation in which $D_0 = D'$. Then R' is a submodule of the finite R_0-module D_0, hence R' is finitely generated.

If R is only semi-prime, let I_1 be a minimal prime of R and let I_2 be the intersection of the remaining primes. Let $R'_v = R'/I'_v$, where $I'_v = I_v \cap R'$. By induction, R'_ν is a finitely generated quotient of R'. If $R' \approx R'_1$ we are done. If not, there is an exact sequence

$$0 \to R' \to R'_1 \times R'_2 \to \varepsilon' \to 0,$$

where $\varepsilon' \approx R'/(I'_1 + I'_2)$ is an R'_1-module whose support has dimension zero, and is therefore a finite k-module. The fact that R' is finitely generated follows easily.

LEMMA 2.5. *Let $\mathcal{O}$ be a discrete valuation ring with field of fractions K. Let A be an $\mathcal{O}$-subalgebra of $M_n(K)$ such that $AK = M_n(K)$. Then A is a finite $\mathcal{O}$-module, or else $A = M_n(K)$.*

Proof. Let t be a generator for the maximal ideal of $\mathcal{O}$. Then $KA = A[t^{-1}]$. Since this is the full matrix ring, $t^n e_{ij} \in A$ for some large n and all indices i, j. If $t^{2n} A \subset M_n(\mathcal{O})$, then A is a submodule of the finite module $t^{-2n} M_n(\mathcal{O})$, and is a finite module. If not, there is a matrix $\alpha = (a_{ij}) \in A$ and a pair i, j of indices, such that $c = t^{2n} a_{ij} \notin \mathcal{O}$. Then

$$\sum_\nu t^n e_{\nu i} \alpha t^n e_{j\nu} = c$$

is in $A \cap K$ but not in $\mathcal{O}$. So $\mathcal{O}[c] = K \subset A$, and $A = M_n(K)$.

We can now prove Propositions 2.1, 2.2. Assume first that R is prime, and let K be the field of fractions of the center Z of R. By dimension theory, K has transcendence degree 1 [9, p. 116]. Therefore $Q = KR$ is isomorphic to a matrix algebra $M_n(K)$ by Tsen's theorem [15]. Since R is finitely generated, we can find a finitely generated subring D' of K such that $R \subset M_n(D')$. Then $Z \subset D'$, hence Z is finitely generated by Lemma 2.4.

To show R finite over Z, let D be the integral closure of Z, and let $R_1 = DR$ and $R' = D'R$:

$$\begin{array}{ccccc} R & \subset & R_1 & \subset & R' \\ \cup & & \cup & & \cup \\ Z & \subset & D & \subset & D'. \end{array}$$

Then R' is a finite D'-module since it is a submodule of $M_n(D')$. To show R_1 finite over D is a local problem on Spec $D = C$. Since we already know it on the open set $C' = \operatorname{Spec} D'$, it suffices to show that for every $p \notin C'$, R_{1p} is finite over D_p. This follows from Lemma 2.5, because D_p is the center of R_{1p}. Thus R is a finite Z-module, because $R \subset R_1$ and each of the inclusions $Z \subset D \subset R_1$ is finite. This proves Proposition 2.1(i) for prime rings, and Proposition 2.1(ii). Assertion (i) of Proposition 2.2 also follows from Proposition 2.1(ii) and Lemma 2.3.

Suppose R is semi-prime. Let I_1 be a minimal prime ideal of R, and let I_2 be the intersection of the remaining minimal primes. Consider the exact diagram

$$\begin{array}{ccccccccc} 0 & \to & R & \to & R_1 \times R_2 & \to & \varepsilon & \to & 0 \\ & & \cup & & \cup & & \uparrow & & \\ 0 & \to & Z & \to & Z_1 \times Z_2 & \to & \delta & \to & 0. \end{array}$$

The vertical arrow is injective because $Z = R \cap (Z_1 \times Z_2)$. Also, $\varepsilon \approx R/(I_1 + I_2)$ is an R_1-module whose support has dimension zero. By induction on the number of minimal primes, Z_i is finitely generated and R_i is a finite Z_i-module $(i = 1, 2)$. Therefore Z is finitely generated by Lemma 2.4, and ε is a finite k-vector space. It follows that $Z_1 \times Z_2$ is a finite Z-module, hence that $R_1 \times R_2$ and R are finite Z-modules. This completes the proof of Proposition 2.1.

Finally, to prove Proposition 2.2(ii), note that the ring of fractions Q of R is a central extension of R'. It follows that $Z' \subset Z$, and so Z' is finitely generated, by Lemma 2.4. One now reduces to the case that Z', Z are integrally closed by passing to their integral closures, and then applies Lemma 2.5 again. These steps are similar to ones we have already taken so we omit them.

We will call the spectrum $C = \operatorname{Spec} A$ of any order a *curve*. The geometry of such a curve can be described easily in terms of the geometry of $\operatorname{Spec} D$. Namely, there will be some non-zero element $t \in D$ so that $A[t^{-1}] = A'$ is isomorphic to the full matrix algebra $M_n(D')$ over $D' = D[t^{-1}]$. The open set $\operatorname{Spec} A'$ of C is then homeomorphic to $\operatorname{Spec} D'$. Since $\operatorname{Spec} D$ has dimension 1, there are finitely many points p of $\operatorname{spec} D$ at which $t = 0$, and the finitely many remaining points of C lie over them. We call the points lying over a given point $p \in \operatorname{Spec} D$ *associated points*. Here is a schematic diagram of the two spectra:

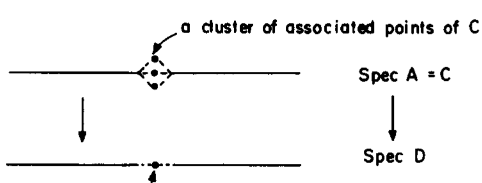

FIGURE 2.6

More information about the maximal ideals lying over a point $p \in \operatorname{Spec} D$ can be obtained by studying an A-invariant lattice: Since $A \subset M_n(K)$, we can view the vector space $V = K^n$ as an A-module. A *lattice* L is a non-zero finitely generated A-submodule of V^r, which generates V^r over K. Since A is a finite module over D, a lattice L will be a projective D-module, obviously of rank rn. Thus if $k(p) \approx k$ denotes the residue field $D/\mathfrak{m}_p$, then $k(p) \otimes_D L = L/\mathfrak{m}_p L = W$ is a vector space over k on which $A/\mathfrak{m}_p A$ acts, and so it determines a representation $p: A \to \operatorname{End}_k W$ of A, of dimension rn.

The isomorphism class of a lattice L is not unique. In fact any A-module M which is a projective D-module of rank n is isomorphic to a lattice in V. Correspondingly, the representation of A on $L/\mathfrak{m}_p L$ is not unique.

EXAMPLE 2.7. Two non-isomorphic lattices. Let $D = k[t]$, and let A be the order

$$A = \begin{bmatrix} D & D \\ tD & D \end{bmatrix}.$$

Let $\{v_1, v_2\}$ be the standard basis of $V = K^2$. The two lattices

$$L = Dv_1 + Dv_2,$$
$$L' = Dt^{-1}v_1 + Dv_2$$

lead to the two non-isomorphic representations $A \to M_2(k)$ defined by

$$\begin{pmatrix} a(t) & b(t) \\ tc(t) & d(t) \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \text{ respectively } \begin{pmatrix} a & 0 \\ c & d \end{pmatrix},$$

where $a = a(0)$, etc.

However, the simple factors of the A-module $W = L/\mathfrak{m}_p L$ are uniquely determined. Let ρ^{ss} denote the semisimple representation associated to $(L/\mathfrak{m}_p L)$. By this we mean the representation corresponding to the graded A-module $\bigoplus W_{i-1}/W_i$, where $W = W_0 \supset \cdots \supset W_l = 0$ is a saturated filtration of W. Every irreducible factor of ρ^{ss} corresponds to one of the points $p_1, ..., p_l$ of $\operatorname{Spec} A$ lying over p. We may write ρ^{ss} symbolically in the form of an unordered l-tuple π of points, with each p_i occurring once for each time it is a simple factor of ρ^{ss}. We have

$$\sum_i \text{pi} \deg p_i = rn = \dim L/\mathfrak{m}_p L. \tag{2.8}$$

PROPOSITION 2.9. *Let A be a D-order, $L \subset V^r$ a lattice for A, and $p \in$ Spec D. The l-tuple π of points corresponding to the representation of A on $L/\mathfrak{m}_p L$ is independent of L, and each point of* Spec *A over p occurs. The l-tuple is determined by the evaluations of certain characteristic polynomials at p.*

LEMMA 2.10. *Let $g(x) \in D[x]$ be the characteristic polynomial of an element $a \in A$. The characteristic polynomial of $\rho(a)$ is $\bar{g}(x)^r$, where $\bar{g}$ denotes the reduction of g module $\mathfrak{m}_p$.*

This lemma is proved by considering the diagram

$$\begin{array}{ccc} A & \to \mathrm{End}_D L & \to \mathrm{End}(L/\mathfrak{m}_p L) \\ \cap & \cap & \\ M_n(K) & \to \mathrm{End}_K(V^r). & \end{array}$$

Proof of Proposition 2.9. It is clear that any point occurring in π lies over p, since $\mathfrak{m}_p A$ goes to zero under the representation. Let $\{p_1,..., p_s\}$ be the points lying over p. Choose $a \in A$ so that the residue of a is 1 at p_1 and is 0 at p_i for $i \neq 1$. If $g(x) = \det(x-a) \in D[x]$ is the characteristic polynomial of a, considered as an element of A, then by the lemma, $\bar{g}(x)^r$ is the characteristic polynomial of $\bar{a} \in \mathrm{End}(L/\mathfrak{m}_p L)$. It follows that $\bar{g}(x)^r = (x-1)^{rm} x^{r(n-m)}$, where m is the product of pi $\deg(p_1)$ and the multiplicity of p_1 in π. Since $\bar{g}(x)^r$ is independent of L, so is the multiplicity of p_1. Finally we note that p_1 must occur in π. For otherwise $m=0$, so $g(x) \in x^m + \mathfrak{m}_p D[x]$. But then by Cayley–Hamilton, we have $a^n \in \mathfrak{m}_p A$, so $a^n \in p_1$, which is a contradiction.

3. ADJOINING TRACES

In this section we discuss adjunction of the coefficients of the characteristic polynomials of elements to our ring R. The customary procedure is to consider a prime ring R as a subring of a ring $M_n(\Lambda)$ of matrices over a commutative domain Λ ($n = \mathrm{pi}\deg R$), and to adjoin elements there. A convenient ring to use is $\bar{K}R \approx M_n(\bar{K})$, where $\bar{K}$ is the algebraic closure of the field of fractions of the center $Z = Z(R)$, but the construction does not really depend on the choice. We denote by $\sigma(r)$ any coefficient of the characteristic polynomial of a matrix $r \in M_n(\Lambda)$. Let

$$\bar{T} = \{\sigma(r) | r \in R \subset M_n(\Lambda)\}. \tag{3.1}$$

The *trace ring* of R is the prime subring $R[\bar{T}]$ of $M_n(\Lambda)$ generated by R and $\bar{T}$. Its center is the k-algebra $k[\bar{T}]$ generated by $\bar{T}$.

The most important properties of the ring diagram

$$\begin{array}{ccc} R & \subset & R[\bar{T}] \\ & & \cup \\ & & k[\bar{T}] \end{array} \tag{3.2}$$

are summarized in the proposition below.

PROPOSITION 3.3. *Let R be a prime affine k-algebra.*

(i) *The ring $k[\bar{T}]$ is a finitely generated k-algebra, and the trace ring $R[\bar{T}]$ is a finite $k[\bar{T}]$-module.*

(ii) *Let $p(x_1,..., x_l)$ be a central polynomial for $n \times n$ matrices without constant term, and let $c = f(\alpha_1,..., \alpha_l)$ be an evaluation of f in R. Then for some integer m, c^m is in the conductor of R in $k[\bar{T}]$, i.e., $c^m R[\bar{T}] \subset R$.*

This proposition reduces formally to the case that R is a ring generated by generic matrices. The first assertion of (i) is due to Procesi [9, pp. 147–150], and the second follows from it by Shirshov's theorem [9, p. 152]. Part (ii) is proved in [13, Theorem 1]: It is known that f vanishes on matrices of rank $<n$ [9, Proposition 1.2, p. 171; 10]. Therefore $R[c^{-1}]$ has no representation of rank $<n$, and is an Azumaya algebra with center $Z[c^{-1}]$. Consequently if $r \in R$, any coefficient $\sigma(r)$ lies in $Z[c^{-1}]$; hence $c^m \sigma(r) \in Z \subset R$ for some m. If we specialize to the case that $R = k\{x, y\}$ is the ring generated by generic matrices $x_1,..., x_l$, y, then we find

$$f(x)^m \det y = g(x, y), \tag{3.4}$$

for some polynomial g. Now it is well known, and easy to see from the definition of characteristic polynomial, that every element a of $k[\bar{T}]$ is a sum of determinants:

$$a = \sum_i \det r_i, \qquad r_i \in R. \tag{3.5}$$

Therefore every element of $R[\bar{T}]$ has the form

$$\beta = \sum_\nu (\det r_\nu)\, s_\nu, \qquad r_\nu, s_\nu \in R. \tag{3.6}$$

Substitution into identity (3.4) shows that

$$c^m \beta = \sum_\nu (c^m \det r_\nu)\, s_\nu = \sum_\nu g(\alpha, r_\nu)\, s_\nu \in R,$$

which proves that $c^m R[\bar{T}] \subset R$, as required.

The trace ring is not completely suitable for our purposes because we need a definition which works for non-prime rings, and also because it is not functorial in R. It turns out that a functorial construction can be obtained in several ways, starting with the trace ring for generic matrices. We will use the one given by Procesi [10].

For any finitely generated k-algebra R, let $R_{(n)}$ denote the quotient of R by the ideal of evaluations of $n \times n$ matrix identities. Thus $R_{(n)}$ is the universal quotient of R of pi degree n, and so it has a presentation $R_{(n)} \approx F/I$ as a quotient of a ring $F = k\{x_1, \ldots, x_l\}$ generated by generic matrices $x_\nu = (x_{ij}^\nu)$. We view F as subring of the matrix ring $M_n(k[x_{ij}^\nu]) = M_n$. It is easily seen that

$$M_n/M_nIM_n \approx M_n(\Lambda), \tag{3.7}$$

where

$$\Lambda = k[x_{ij}^\nu]/\mathscr{I},$$

$\mathscr{I}$ being the ideal generated by the entries of the matrices in I. The situation can be summed up in the diagram

$$\begin{array}{ccccc} k\{x\} = & F & \subset & M_n(k[x_{ij}^\nu]) \\ & \downarrow & & \downarrow \\ R \twoheadrightarrow R_{(n)} \approx & F/I & \to & M_n(\Lambda). \end{array} \tag{3.8}$$

DEFINITION 3.9. $R[T_n] = R[T_n(R)]$ denotes the subring of the ring $M_n(\Lambda)$ of (3.8) generated by R and $\{\sigma(r) \mid r \in R\}$, and $k[T_n(R)]$ denotes the commutative subring generated by $\{\sigma(r)\}$. The canonical map $R \to R[T_n]$ is denoted by $\tau = \tau_n(R)$.

We will show (3.13) that this construction is canonical and functorial in R. For the present, it is clear that when $R = F$ is a ring of generic matrices, then $R[T_n]$ is isomorphic to the trace ring (3.2). Also, if $R = F/I$ is a quotient of F, then $R[T_n(R)]$ is a quotient of $F[T_n(F)]$, and $k[T_n(R)]$ is a quotient of $k[T_n(F)]$. Therefore Proposition 3.3 yields:

PROPOSITION 3.10. *Let R be any finitely generated k-algebra. Then*

(i) *The ring $k[T_n(R)]$ is a finitely generated k-algebra, and $R[T_n(R)]$ is a finite module over $k[T_n(R)]$.*

(ii) *Let c be an evaluation in R of a central polynomial for $n \times n$ matrices, without constant term. Then there is an integer m and an R-linear map $R[T_n] \to R$, so that the composition $R \to^{\tau} R[T_n] \to R$ is multiplication by c^m.*

Let $\mathscr{S}$ be the category whose objects are rings equipped with embeddings into $n \times n$ matrix rings over commutative rings: $S \hookrightarrow M_n(E)$, subject to the condition that $\sigma(s) \in S$ for every s in S. Morphisms in $\mathscr{S}$ are diagrams

$$\begin{array}{ccc} M_n(E) & \xrightarrow{\bar{\psi}} & M_n(E') \\ \uparrow & & \uparrow \\ S & \longrightarrow & S' \end{array} \tag{3.11}$$

induced by ring maps $E \to E'$. It is clear that $\bar{\psi}(\sigma(s)) = \sigma(\bar{\psi}(s))$.

PROPOSITION 3.12. *Let $S \hookrightarrow M_n(E)$ be in $\mathscr{S}$, and let $R \to^{\varphi} S$ be any ring homomorphism. If Λ denotes the ring (3.7), there exists a unique map $\Lambda \to^{\psi} E$, so that the induced map $M_n(\Lambda) \to^{\psi} M_n(E)$ restricts to a map $R[T_n] \to S$, and makes the diagram below commutative:*

$$\begin{array}{ccccc} R[T_n] & & & \Lambda & \\ {\scriptstyle\tau}\uparrow & \searrow & & & \searrow{\scriptstyle\psi} \\ R & \longrightarrow S & & & E \end{array} \tag{3.13}$$

Remark (3.14). Proposition 3.12 asserts that the construction assigning to R the object $R[T_n] \hookrightarrow M_n(\Lambda)$ of $\mathscr{S}$ is the left adjoint of the functor from $\mathscr{S}$ to the category of all rings which forgets the matrix structure. Therefore it is functorial.

Proof of Proposition 3.12. The map $\lambda: F \to R_{(n)} \to^{\varphi} S \hookrightarrow M_n(E)$ induces a unique map $k[x_{ij}^{\nu}] \to^{\psi_0} E$, sending x_{ij}^{ν} to the (i, j) entry of the matrix $\lambda((x_\nu))$. If $u \in I$, then $\lambda(u)$ is the zero matrix. Thus $\psi_0(u_{ij}) = 0$, so ψ_0 factors through Λ, and induces a map $\Lambda \to^{\psi} E$, which has the required property.

PROPOSITION 3.15. *If R is Azumaya, of pi degree n, then the map $R \to^{\tau} R[T_n]$ is an isomorphism.*

Proof. For any R, there exist evaluations of central polynomials $c_1, \ldots, c_l$ so that the zero set of $c_1, \ldots, c_l$ is $(\operatorname{Spec} R - \operatorname{Spec}_n R)$ [9, p. 178]. For Azumaya algebras this set is empty, so we can write $1 = \sum c_i r_i$. Thus for m large, $\sigma(r) = \sigma(r)((\sum c_i r_i)^{ml}$ is contained in the subring of $M_n(D)$ generated by R, by Proposition 3.3(ii). This shows that τ is surjective. To see that it is 1–1 recall that R has a faithfully flat extension $R \to M_n(E)$, where E is commutative. Apply Proposition 3.12 with $S = R$ and $\varphi =$ identity to obtain a section for τ.

LEMMA 3.16. *If R is a prime ring of pi degree n then the map $R \to^{\tau} R[T_n]$ is an injection, and the trace ring $R[\bar{T}]$ (3.2) is a factor ring of $R[T_n]$.*

Proof. If K is the fraction field of the center of R, then RK is central simple, hence an Azumaya algebra. We have

$$\begin{tikzcd} R[T_n] \arrow[r] & RK[T_n] \\ R \arrow[u, "\tau"] \arrow[r, hook] & RK \arrow[u, "\wr"'] \end{tikzcd}$$

The right arrow is an isomorphism by Remark 3.14. This shows the first part of the lemma.

To see that $R[\bar{T}]$ is a factor ring of $R[T_n]$, we apply (3.11) with $\varphi: R \to R[\bar{T}]$ and $R[\bar{T}] \hookrightarrow M_n(\bar{K})$ playing the role of $S \hookrightarrow M_n(E)$.

EXAMPLE 3.17. A prime ring R of pi degree 2 such that $R[T_2]$ is not prime. Let

$$F = k\{z, w\}, \qquad R = k\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \right\}$$

be a presentation of R. Then $I = \ker F = (z^2, w^2)$. We have $D = k[z_{ij}, w_{ij}]/(z_{11}z_{11} + z_{12}z_{21}, z_{21}z_{11} + z_{22}z_{21},...)$, where the dots represent the other six entries of z^2 and w^2. Because it is linear, $\operatorname{tr} z$ is not zero in D, and so it is non-zero in $R[T_2] \subseteq M_2(D)$. However $(\operatorname{tr} z)^4 = 0$ in D, and prime rings do not have nilpotent central elements.

The remainder of this section is devoted to describing the spectra of the rings $k[T_n]$ and $R[T_n]$. Procesi [10] shows that $\operatorname{Spec} k[T_n]$ is naturally identified with the space $SS_n(R)$ of semisimple representations of R of rank n. We review his result, and also prove that points of $\operatorname{Spec} R[T_n]$ are in 1–1 correspondence with pairs consisting of a semisimple representation together with a simple factor.

It is natural to associate a semisimple representation $\rho = \rho_1 \oplus \cdots \oplus \rho_l$ of R to the unordered l-tuple of (not necessarily distinct) points $\pi = (p_1, ..., p_l)$ of $X = \operatorname{Spec} R$ such that

$$\mathfrak{m}_{p_i} = \ker(\rho_i : R \twoheadrightarrow \operatorname{End}_k W_i). \tag{3.18}$$

Then $\operatorname{End}_k W_i \approx R/\mathfrak{m}_{p_i}$, and ρ has rank n if and only if

$$\text{degree } \pi = \sum_i \text{pi deg } p_i = n. \tag{3.19}$$

THEOREM 3.20. *Let R be a finitely generated k-algebra. With the above notation,*

(i) Spec $k[T_n(R)] \approx SS_n(R) \approx \{\pi \mid \textit{degree } \pi = n\}$.

(ii) Spec $R[T_n(R)] \approx \{(\pi, p) \mid \textit{degree } \pi = n, \textit{ and } p \in \pi\}$.

(iii) *The subrings R, $k[T_n]$ induce the following maps of spectra*:

$$\begin{array}{ccc} \operatorname{Spec} R \leftarrow \operatorname{Spec} R[T_n] & & p \leftsquigarrow (\pi, p) \\ \downarrow & \textit{is given by} & \wr \\ \operatorname{Spec} k[T_n] & & \pi. \end{array}$$

Proof. We first consider $R[T_n]$. Let (π, p) be as in the theorem, and say that $p = p_1$. With notation as in (3.18), the homomorphism $\rho: R \to \prod \operatorname{End}_k W_i \subset \operatorname{End}_k(\bigoplus W_i) \approx M_n(k)$ extends to $R[T_n]$, by Proposition 3.12:

$$\begin{array}{ccccc} R[T_n] & \to & \prod \operatorname{End}_k W_i & \subset & M_n(k) \\ \cup & & \cup & & \cup \\ k[T_n] & \to & k & = & k. \end{array} \tag{3.21}$$

Projection to the first factor determines a surjective homomorphism

$$\varphi: R[T_n] \to \operatorname{End}_k(W_1), \tag{3.22}$$

hence a maximal ideal $\mathfrak{m}_{(\pi,p)} = \ker \varphi$ of $R[T_n]$, whose inverse image in R is $\tau^{-1}(\mathfrak{m}_{(\pi,p)}) = \mathfrak{m}_p$.

To prove Theorem 3.20(ii), we have to show that this correspondence $(\pi, p) \to \mathfrak{m}_{(\pi,p)}$ is bijective. First, it is injective: Suppose $\mathfrak{m}_{(\pi,p)} = \mathfrak{m}_{(\pi',p')}$. Then $\mathfrak{m}_p = \tau^{-1}(\mathfrak{m}_{(\pi,p)}) = \mathfrak{m}_{p'}$; hence $p = p'$. Let $q \in \pi$. By the Chinese remainder theorem, we may choose $r \in R$ taking the value 1 at q and vanishing at all other points of π and π'. Then the characteristic polynomial of $\rho(r)$ has the form $(x-1)^{de} x^{n-de}$, where e is the multiplicity of q in π and $d = \text{pi} \deg q$. By construction of φ (3.22), $\varphi(\sigma) = \rho(\sigma)$ for any coefficient σ of the characteristic polynomial of r in $R[T_n]$, and the same is true for the semisimple representation ρ' associated to π'. Therefore $e = e'$.

We will prove surjectivity first for a ring R of generic $n \times n$ matrices, for which $R[T_n] = R[\bar{T}]$ is the trace ring. Let $p' \in \operatorname{Spec} R[\bar{T}]$ be a point lying over $p_1 \in \operatorname{Spec} R$. Since $R[\bar{T}]$ is a central extension of R, the pi degree d of p' is at most n. In case $d = n$, we obtain an irreducible representation of R of rank n,

$$\rho_1 : R \to R[\bar{T}]/\mathfrak{m}_{p'} \approx M_\alpha(k),$$

and $\mathfrak{m}_{p'}$ is the maximal ideal associated to the pair $((p_1), p_1)$. In general, we have to construct a semisimple representation of rank n of R containing ρ as a simple factor. We do this by choosing an appropriate curve, i.e., central homomorphism from $R[\bar{T}]$ to an order A (cf. Section 5 for a general treatment of such curves).

We know (Proposition 3.3) that $R[\bar{T}]$ is a finite module over its center $k[\bar{T}]$, and that $k[\bar{T}]$ is finitely generated, hence integral over a polynomial subring. The "going down" theorem [13, Theorem 3] implies the existence of a 1-dimensional irreducible closed subscheme $Y \subset \operatorname{Spec} R[\bar{T}]$ containing p', which we may assume meets the open set $\operatorname{Spec}_n R[\bar{T}]$ of points of pi degree n (see, for example, the proof of [13, Lemma 5]). Let $\mathfrak{p}'$ be the prime ideal of $R[\bar{T}]$ corresponding to Y. Then $R[\bar{T}]/\mathfrak{p}' = B$ is a one-dimensional prime ring. We replace B by $A = DB$, where D is the integral closure of the center $Z(B)$. Then Proposition 2.1 shows that A is an order over D in $M_n(K)$. Since $p' \in Y$ and $C = \operatorname{Spec} A$ is finite over Y, there is a point $p'' \in C$ lying over p'. The choice of a lattice L yields the required representation ρ of R, by Proposition 2.9. If $\mathfrak{p} = \mathfrak{m}_{p''} \cap D$, then P is the semisimple representation associated to the R-module structure on $L/\mathfrak{p}L$.

Now let R be any finitely generated k-algebra, and revert to the notation of (3.8): $R_{(n)} = F/I$. Then $\operatorname{Spec} R$ is a closed subset of $\operatorname{Spec} F$. Since $R[T_n]$ is a quotient of $F[T_n]$, and the surjectivity is proved for F, it suffices to show that if (π, p) is a pair as above for the ring F, then $\mathfrak{m}_{(\pi,p)} \supset \ker(F[T_n] \to R[T_n])$ if and only if every point $q \in \pi$ lies in $\operatorname{Spec} R$. This fact is contained in the following lemma:

LEMMA 3.23. *With the above notation, the following are equivalent*:

(i) $\sigma(I) \subset \mathfrak{m}_{(\pi,p)}$,

(ii) $I \subset \bigcap \mathfrak{m}_q, q \in \pi$,

(iii) $\sigma(r) - \sigma(s) \in \mathfrak{m}_{(\pi,p)}$ *whenever* $r - s \in I$.

(iv) $\mathfrak{m}_{(\pi,p)} \supset \ker(F[T_n] \to R[T_n])$.

Proof. (i) $\Rightarrow$ (ii). If $r \in I$, then by (i) its characteristic polynomial of $\rho(r)$ is x^n. Thus I is nilpotent modulo $\mathfrak{m}_q$ for each $q \in \pi$. This implies (ii), (ii) $\Rightarrow$ (iii): If $r - s \in I$, then (ii) states that $r(q) = s(q)$ for each $q \in \pi$. Thus $\rho(r)$ and $\rho(s)$ have the same characteristic polynomial, and hence $\sigma(r) - \sigma(s)$ will go to zero in the extension of ρ to $R[T_n]$. (iii) $\Rightarrow$ (iv): Property (iii) implies that (I) is nilpotent. Since ρ is semisimple, $\rho(I) = 0$. We can extend $\rho: F \to M_n(k)$ in the canonical way to $\rho': M_n(k[x_{ij}]) \to M_n(k)$. Then $\rho'(I) = \rho(I) = 0$. Therefore $\ker(F[T_n] \to R[T_n]) \subset M_n I M_n$ maps to zero when ρ is extended to $F[T_n] \subset M_n(k[x_{ij}^{\nu}])$. This implies (iv). The remaining implication (iv) $\Rightarrow$ (i) is clear. This completes the proof of the lemma and of Theorem 3.20(ii).

To prove Theorem 3.20(i), note first that $\operatorname{Spec} R[T_n]$ maps onto $\operatorname{Spec} k[T_n]$, because $R[T_n]$ is integral over $k[T_n]$. Moreover we have already seen that the values $\rho(\sigma(r))$ determine the semisimple representation ρ.

Therefore (π, p) and (π', p') have the same image in $\operatorname{Spec} k[T_n]$ if and only if $\pi = \pi'$. Assertion (iii) of the theorem is also contained in what has been shown.

Representations over arbitrary fields can be handled similarly, provided finite field extensions are permitted.

PROPOSITION 3.24. *Let R be a ring, and $T = T_n(R)$. Let $k[T] \to^{\pi} K$ be a homomorphism, where K is a field. There exists a finite field extension K' of K and a semisimple representation $R[T] \to^{\sigma} \bigoplus_{\nu} M_{i_\nu}(K')$ inducing $k[T] \to^{\pi'} L'$. The representation σ is unique up to isomorphism.*

We omit the proof of this proposition.

4. THE CORRESPONDENCE INDUCED BY A RING HOMOMORPHISM

A ring homomorphism $R \to^{\varphi} S$ will not, in general, induce a single-valued map $\operatorname{Spec} S = Y \to X = \operatorname{Spec} R$, because if $\mathfrak{m}$ is a maximal ideal of S, then $\varphi^{-1}(\mathfrak{m})$ need not be maximal (or prime) in R. Instead of a single-valued map, one obtains a finite-valued *correspondence*

$$Y \xrightarrow{f} X \tag{4.1}$$

as follows: Let $y \in Y$, and let $S \to^{\rho} M_n(k)$ be the associated irreducible representation. We form the composed representation $R \to^{\rho\varphi} M_n(k)$. The semisimple representation associated to $\rho\varphi$ is determined by a finite l-tuple $(x_1, ..., x_l)$ of points of X (Proposition 3.24), and we define

$$f(y) = (x_1, ..., x_l).$$

Thus

$$\deg y = \sum_1^l \deg x_i, \tag{4.2}$$

and in particular,

> Every point of Y corresponds to a finite, nonempty family of points of X. The points x which occur in $f(y)$ are those such that $\mathfrak{m}_x \supset \varphi^{-1}(\mathfrak{m}_y)$. (4.3)

EXAMPLE 4.4. We return to Example 1.5: $R = k[x]$, $S = M_2(k[u])$, and $\varphi(x) = \alpha$. The irreducible representations ρ of S are given by evaluation maps $u \rightsquigarrow u_0 \in k$. Let $\alpha_0 = \alpha(u_0)$. The representation $\rho\varphi$ of $R = k[x]$ is therefore $x \rightsquigarrow \alpha_0$, and its associated semisimple representation is

$$x \rightsquigarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_i are the eigenvalues of α_0. Thus the point $u = u_0$ of Y corresponds to the point pair (λ_1, λ_2) of points in X, and the graph of the correspondence is given by the Cayley–Hamilton equation, as asserted in Example 1.5.

We will often refer to correspondence (4.1) as a *morphism* $Y \to^f X$. Note that the inverse correspondence is given by

$$\begin{aligned} &f^{-1}(x) = \{y \in Y \mid f(y) \text{ contains } x\}, \text{ and if } A \subset X, \\ &f^{-1}(A) = \{y \in Y \mid f(y) \cap A \neq \varnothing\}. \end{aligned} \tag{4.5}$$

PROPOSITION 4.6. (i) *Let $R \to^\varphi S \to^\psi T$ be homomorphisms with associated correspondences $Z \to^g Y \to^f X$. Then the correspondence determined by $\psi\varphi$ is the composition fg. The same is true for the inverse correspondences.*

(ii) *If φ is an extension then f is single valued.*

(iii) *If φ is surjective, f is single valued and maps Y homeomorphically to the closed subset $V(\ker \varphi)$ of X.*

(iv) *If φ is a central localization: $S = R[c^{-1}]$, then f is single valued and maps Y homeomorphically to the open subset $X - V(c)$ of X.*

(v) *The correspondence f is continuous: If $U \subset X$ is open then $f^{-1}(U) = \{y \in Y \mid f(y) \cap U \neq \varnothing\}$ is open.*

(vi) *If φ is an injective homomorphism, the image $f(Y)$ is dense in X.*

The proofs of these assertions are routine, and we will only include a proof of (vi): Let $I \subset R$ be the ideal of elements vanishing on $f(Y)$, let $y \in Y$ and let $S \to^\rho M_n(k)$ be the representation associated to y. Then $\alpha \in \mathfrak{m}_x$ for all $x \in f(y)$, which means that α maps to zero in the semisimple representation associated to $\rho\varphi$. Therefore $\rho\varphi(\alpha)$ is nilpotent; hence $\varphi(\alpha)^n \in \mathfrak{m}_y$. Since the exponent n is bounded by the pi degree of S, it follows that for large n, $\varphi(I)^n$ is in the intersection of all maximal ideals of S, and (since φ is injective) that I is a nilpotent ideal of R. Therefore $V(I) = X$, and $f(Y)$ is dense in X.

With regard to Proposition 4.6(v), let $U = X - V(I)$. Then one shows easily that $f^{-1}(U)$ is the complement of the closed set in Y defined by the ideal SIS. It is not true, however, that the inverse images of closed sets are closed. The classical definition of continuity is thus seen to be the correct one.

EXAMPLE 4.7. A closed set whose inverse image is not closed. Let $A = k[u]$,

$$R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} A & uA \\ A & A \end{pmatrix}. \tag{4.8}$$

The spectra of R and S are illustrated below:

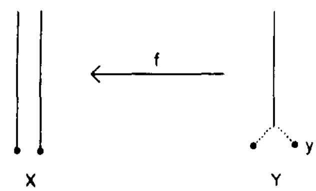

FIGURE 4.9

Let $C \subset X$ be the closed set defined by the ideal $Re_{11}R$. Then $f^{-1}(C)$ is the complement of the point y which is associated with the maximal ideal

$$\mathfrak{m}_y = \begin{pmatrix} A & uA \\ A & uA \end{pmatrix}. \tag{4.10}$$

Thus $f^{-1}(C)$ is dense but not closed.

We recall that a *constructible* set is one which is a finite union of locally closed sets, i.e., of sets of the form $C \cap U$, where C is closed and U is open. These are the sets which can be obtained from the family of open sets by complementation, finite unions and finite intersections.

PROPOSITION 4.11. *The correspondences f and f^{-1} carry constructible sets to constructible sets.*

Proof. We use the following standard fact:

LEMMA 4.12. *Let $Y = \operatorname{Spec} S$. Then Y can be partitioned by finitely many locally closed subsets of the form $U_v = \operatorname{Spec} A_v$, where each A_v is a prime Azumaya algebra, and $S \to A_v$ is a composition of a surjection, followed by a central localization.*

Let H be a constructible subset of X. To show $f^{-1}(H)$ is constructible, it suffices to show that $f^{-1}(H) \cap U_v$ is constructible for each U_v of a partition (Lemma 4.12). So by using Proposition 4.6 we may replace S by A_v, i.e., suppose that S is Azumaya. If H is constructible in Y, then $f(H) = \bigcup f(H \cap U_v)$, and a similar reasoning applies. Again we may assume S is Azumaya. We do so.

Let $Z = Z(S)$, let $R' = R[Z]$ be the subring of S generated by $\varphi(R)$ and Z, and let $X' = \operatorname{Spec} R'$. Then the morphism f fits into a diagram

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[dr, "p"'] & & Y; \arrow[ll, "h"'] \arrow[dl, "q"] \\ & & \operatorname{Spec} Z & \end{tikzcd} \qquad f = gh. \tag{4.13}$$

Since S is an Azumaya algebra with center Z, $q = ph$ is a homeomorphism. Also it is easy to see that $p^{-1} = hq^{-1}$, though this fact is not completely formal. For, let $z \in \operatorname{Spec} Z$. By the Artin–Rees lemma, $\mathfrak{m}_z R' \supset \mathfrak{m}_z^n S \cap R'$ for large enough n. By definition of p, $x \in p^{-1}(z)$ if and only if $\mathfrak{m}_x \supset \mathfrak{m}_z R'$. Then $\mathfrak{m}_x \supset \mathfrak{m}_z^n S \cap R'$, and since $\mathfrak{m}_x$ is maximal, this implies that $\mathfrak{m}_x \supset \mathfrak{m}_z S \cap R'$. Let $y = q^{-1}(z)$. Then $\mathfrak{m}_z S = \mathfrak{m}_y$, so $\mathfrak{m}_x \supset \mathfrak{m}_y \cap R'$, and $x \in h(y)$. Thus $p^{-1}(z) \subset hq^{-1}(z)$. Conversely, if $\mathfrak{m}_x \supset \mathfrak{m}_y \cap R'$, then $\mathfrak{m}_x \supset \mathfrak{m}_z R'$, so $x \in h(y)$ implies $x \in p^{-1}(x)$.

It follows from these facts that we need to prove the proposition only for the maps g and p, hence that we are reduced to the two cases:

(a) φ is central,

(b) R is commutative and S is a finite R-algebra.

In case (a) we partition $X = \operatorname{Spec} R$ as in Lemma 4.12 using some homomorphisms $R \to B_\nu$. Then S is partitioned by $\operatorname{Spec}(B_\nu \otimes_R S)$, and we are thus reduced to the case that R, S are both Azumaya. Then the proposition is reduced to the homomorphism of commutative rings $Z(E) \to Z(S)$. We refer to the literature for this case [5, p. 94, exc. 3.19]. In case (b), we reduce as before to the case that S is Azumaya, and replace φ by the homomorphism $R \to Z(S)$, again reducing to the commutative case.

COROLLARY 4.14. *The fiber $f^{-1}(x)$ of a morphism $Y \xrightarrow{f} X$ is a constructible set.*

The argument using the Artin–Rees lemma given in the proof of Proposition 4.11 can be extended to prove the following special case of Corollary 7.2.

PROPOSITION 4.15. *Suppose that R is a subring of S, and that both are finite algebras over a commutative ring A. Then the correspondence $Y \xrightarrow{f} X$ has finite fibers and is surjective.*

PROPOSITION 4.16. *Let $Y \xrightarrow{f} X$ be a morphism* (4.1), *and let $V \subset X$ be an irreducible closed set such that $f(Y) \cap V$ is dense in V. Then Y contains an irreducible closed set W such that $f(W) \cap V$ is dense in V, and* $\dim V = \dim W$.

Proof. We partition Y as in Lemma 4.12, and reduce by Proposition 4.6 to the case that S is Azumaya. Then we follow the proof of Lemma 4.12 to construct a diagram (4.13), and are reduced to considering separately the case of a central homomorphism g and the morphism h. The first reduces to commutative algebra as before, and for the second we apply the going up theorem (Proposition 1.10) to conclude that if V' is closed in X' then $p(V') = W'$ is a closed subset of Spec Z of the same dimension. Therefore $W = q^{-1}(W')$ is the required subset of Y.

5. Curves in Spec R

Let R be a ring. For any $V \subset X = \text{Spec } R$ and any integer d, the symbol V_d will denote the (locally closed) subset V of points of pi degree d. By *curve in* X we mean a central homomorphism $R \to^{\varphi} A$ from R to an order over a Dedekind domain D. Such a homomorphism will induce a single-valued map of spectra

$$\text{Spec } A = C \xrightarrow{f} X, \tag{5.1}$$

which is the reason for the terminology. We can view the curve in two ways: either as a one-dimensional family of points of X, or by choosing an A-lattice L, as a one-parameter family of representations of R, all but a finite number of the representations being irreducible. In this interpretation, the associated semisimple representations are independent of the choice of L (Proposition 2.9). Therefore the curve (5.1) defines a uniquely determined map

$$\text{Spec } D \to SS_n(R), \qquad n = \text{pi deg } A, \tag{5.2}$$

having the property that a non-empty open set in Spec D (i.e., the complement of a finite set) maps to the set of irreducible representations of pi degree n (see Theorem 3.20(i)). This set is homeomorphic to $X_n = \text{Spec}_n R$ [10].

The map (5.2) can be described in terms of the trace rings introduced in Section 3: Since A is an order, the characteristic polynomials of any $\alpha \in A$ lie in D. Therefore A is equal to the trace ring $A[\bar{T}]$ (3.1) so (Lemma 3.16) the inclusion of A into $A[T_n(A)]$ splits:

$$A \rightleftarrows A[T_n(A)]. \tag{5.3}$$

The map (5.1) extends to a map $R[T_n(R)] \to A[T_n(A)]$, and by (5.3) the curve φ extends to $R[T_n(R)]$:

$$\begin{array}{c} R \to R[T_n(R)] \to A \\ \cup \\ k[T_n(R)] \to D. \end{array} \tag{5.4}$$

The bottom arrow of this diagram is the one which defines the map (5.2). In down-to-earth terms, it is obtained by sending a coefficient of the $n \times n$ characteristic polynomial of $\alpha \in R$ to the corresponding coefficient of $\varphi(\alpha)$ in D.

Suppose that we begin with any homomorphism $k[T_n(R)] \to D$ such that the associated map (5.2) sends a non-empty open subset U of $\operatorname{Spec} D$ to $\operatorname{Spec}_n R$. We may suppose that $U = \operatorname{Spec} D'$, where $D' = D[1/t]$ for some non-zero $t \in D$. Since $R[T_m]$ is a finite $k[T_m]$-module which is Azumaya at all points of $\operatorname{Spec}_n R$, the ring $A' = D' \otimes_{k[T_m]} R[T_m]$ is an Azumaya algebra. Since the traces generate the center of an Azumaya algebra, D' is the center. Let A be the subring of A' generated by the images of R and D. It follows from (5.4) that A is a D-order, and of course the homomorphism $R \to A$ is central. This shows that the map (5.2) determines the curve (5.1), and so we have the following proposition:

PROPOSITION 5.5. *Diagram* (5.4) *establishes a* 1–1 *correspondence between*

(i) *curves in* $\operatorname{Spec} R$, *of pi degree* n,

(ii) *curves in* $\operatorname{Spec} R[T_n(R)]$, *of pi degree* n,

(iii) *homomorphisms* $k[T_n(R)] \to D$, *where* D *is a Dedekind domain, such that the associated map of spectra sends a nonempty open set of* $\operatorname{Spec} R$ *to* $\operatorname{Spec}_n R$.

COROLLARY 5.6. *Suppose that* R *is a prime ring of pi degree* n. *There is a* 1–1 *correspondence between curves of pi degree* n *in* $\operatorname{Spec} R$ *and in* $\operatorname{Spec} R[\bar{T}]$.

PROPOSITION 5.7. *Let* D *be a Dedekind domain with quotient field* K, *and let* $R \to^{\lambda} M_n(K)$ *be any ring homomorphism. The following are equivalent*:

(i) *The subring* $R[D]$ *of* $M_n(K)$ *generated by* $\lambda(R)$ *and* D *is a finitely generted module over* D.

(ii) *If* $R[T_n(R)] \to^{\bar{\lambda}} M_n(K)$ *is the extension of* λ (*Lemma* 3.16), *then* $\bar{\lambda}(k[T_n(R)]) \subseteq D$.

Proof. (i) $\Rightarrow$ (ii). If $\alpha \in \lambda(R)$, then it satisfies some polynomial in $D[x]$, so its minimum polynomial is in $D[x]$. Hence all its eigenvalues are integral over $D[x]$, so its characteristic polynomial is in $D[x]$.

(ii) $\Rightarrow$ (i). We have $\lambda(R) \subseteq \bar{\lambda}(R[T_n])[D]$, and the latter is a finitely generated D-module by Shirshov's theorem [9, 14].

We now want to study the Zariski topology on $X = \operatorname{Spec} R$. Since the topology of a curve $C = \operatorname{Spec} A$ is easily understood and a curve in X is a continuous map (5.1) $C \to X$, it is natural to use curves for this purpose.

Let V be a constructible subset of X. Its *boundary* is the set $B = \bar{V} - V$, where $\bar{V}$ is the Zariski closure of V. Clearly, B is a constructible set.

DEFINITION 5.8. Let V be a constructible set in X. A point $p \in X - V$ is an *accessible* point of V if there is a curve $C \xrightarrow{f} X$ passing through p, such that a non-empty open subset C' of C maps to V. A boundary point p is called *inaccessible* if it is not accessible. Two points $p_1, p_2 \in X - V$ are *associated* boundary points if for some such curve, $p_i = f(c_i)$ and c_1, c_2 are associated points of C (see Section 2).

Note that accessible boundary points are in the boundary of V, because C' is dense in C. But the points associated to a given accessible point depend on the curve C, and p may be associated to infinitely many other points.

EXAMPLES 5.9. (i) Let $R = k\{x, y\}$, where x and y are generic 2×2 matrices. The boundary of the open set $V = X_2$ is X_1. All points of X_1 are accessible, and any pair of points $p_1, p_2 \in X_1$ is associated. For, given p_i, let $\mathfrak{m}_i \subset R$ be the corresponding maximal ideal, and let $a_i, b_i \in k$ denote the residues of x, y, respectively, modulo $\mathfrak{m}_i$. Let A be the image of the homomorphism $R \to M_2(k[t])$ defined by

$$x \rightsquigarrow \begin{pmatrix} a_1 & 1 \\ & a_2 \end{pmatrix}, \qquad y \rightsquigarrow \begin{pmatrix} b_1 & \\ t & b_2 \end{pmatrix}.$$

Then A is an order over $k[t]$, the map $R \to A$ is a curve in X, and p_1, p_2 are the points lying over $\{t = 0\}$.

(ii) Let $R = k\{x, y\}$, where

$$x = \begin{pmatrix} u & 1 \\ & \bar{u} \end{pmatrix}, \qquad y = \begin{pmatrix} & \\ v & \end{pmatrix},$$

u and v being variables and $\bar{u} = u^{-1}$. Let $V = X_2$ as before. The ring R is prime, and therefore $\bar{V} = X$. The point defined by $x = y = 0$ is an inaccessible boundary point of V.

PROPOSITION 5.10. *Let R be a ring finite over a commutative affine ring Z, and let V be a constructible subset of $X = \operatorname{Spec} R$. Then every boundary point p of V is accessible.*

Proof. This is a consequence of the "going down" theorem [13, Theorem 3]. Elementary topological reasoning shows that it suffices to treat the case that V is an irreducible locally closed set. Then we may replace R by $\bar{R} = R/I$, where I is the ideal of $\bar{V}$. Since V is assumed irreducible, so is $\bar{V}$. Therefore I is a prime ideal. Thus we may assume that R is a prime ring of pi degree n, and that V is a non-empty open subset of X. We can now use the construction of [13, Lemma 5] to find an irreducible one-dimensional subscheme Y containing p and meeting X_n. This Y is the image of a curve, by Proposition 2.1.

PROPOSITION 5.11. *Let R be a prime ring of pi degree n, and let $R' = R[\bar{T}]$ be its trace ring* (3.1). *Let V be a constructible subset of $X_n = \operatorname{Spec}_n R$. The accessible boundary points of V are the boundary points which are in the image of the map* $\operatorname{Spec} R' = X' \to X$.

Proof. We know by Proposition 4.11 that V corresponds to a constructible set V' in X'. By Propositions 5.10 and 3.3, every boundary point of V' is accessible. Therefore the image of X' consists of accessible points. The converse follows from Corollary 5.6.

COROLLARY 5.12. *Let V be a constructible subset of $X = \operatorname{Spec} R$. The accessible boundary points of V form a constructible set.*

By Proposition 4.11, the image of a constructible set is constructible, so the corollary follows from Proposition 5.11 if R is a prime and V is open in X_n. The general case can be deduced by elementary topology.

THEOREM 5.13. *Let V be a constructible subset of $X = \operatorname{Spec} R$. The accessible points of V are dense in the boundary $B = \bar{V} - V$.*

Proof. We first treat the case that R is irreducible and V is an open subset of X_n. Let V' denote the corresponding subset of $X' = \operatorname{Spec} R'$, where $R' = R[\bar{T}]$ is the trace ring. By Proposition 5.10, we have to show that the image of $B' = X' - V'$ is dense in $B = X - V$. Since X'_n and X_n are homeomorphic, we know that $X'_n - V \approx X_n - V$. So it suffices to show that the image of the rest of B', which is $X'_1 \cup \cdots \cup X'_{n-1}$, is dense in $X_1 \cup \cdots \cup X_{n-1}$. In other words, we may assume that $V = X_n$ and $B = X_1 \cup \cdots \cup X_{n-1}$.

By Proposition 3.3(ii) we can find an ideal C of R, contained in the conductor of R in R', such that the variety $V_R(C) \approx \operatorname{Spec} R/C$ defined by C in X is B. Also, C is an ideal in R', and $B' = V_{R'}(C) \approx \operatorname{Spec} R'/C$. Since $R/C \subset R'/C'$, the image of B' in B is dense.

Now let R and V be arbitrary, and let $p \in B$. We have to show that p is in the closure of the set of accessible points. Any constructible set V contains an irreducible locally closed set U which is open in $\bar{V}$, and we may suppose $U \subset X_d$ for some d. Let $U' = V - U$. Then $p \in \bar{U}'$ or $p \in \bar{U}$. In the first case, we replace V by U' and use Noetherian induction. Note that accessible boundary points of U' are accessible boundary points of V. Suppose next we are in the second case. Let $\bar{R} = R/I$, where I is the ideal of $\bar{U}$. This is a prime ring of pi degree d, and $U \subset \operatorname{Spec} \bar{R}$. Therefore the accessible boundary points of U are dense, by what has already been shown. Let the set of these points be Y, so that Y is a constructible (Corollary 5.12) subset of $(\bar{U} - U)$, and $p \in Y = (\bar{U} - U)$. Let $W' = V \cap Y$ and $W = Y - W'$. Then $p \in \bar{W}'$ or $p \in \bar{W}$. In the first case, we replace V by W' and proceed by Noetherian induction. In the second case, we note that every point of W is an accessible boundary point of U, but is not in V. Since $U \subset V$, every point of W is an accessible boundary point of V. Thus in this last case p is in the closure of the set of accessible points, as was to be shown.

As a consequence of Corollary 5.12, we can describe the Zariski topology on $\operatorname{Spec} R$ in terms of curves.

COROLLARY 5.14. *A subset V of $X = \operatorname{Spec} R$ is closed if and only if*

(i) *$V_d = V \cap X_d$ is closed in the algebraic variety X_d for each d (or else V is constructible), and*

(ii) *$f^{-1}(V)$ is closed in C for every curve $C \xrightarrow{f} X$.*

In fact, condition (i) implies that V is constructible. If V is not closed, there is an accessible point x in its boundary $B = \bar{V} - V$, by Corollary 5.12. Let $C \xrightarrow{f} X$ be a curve passing through x, say $x = f(p)$, and such that a non-empty open set $C' \subset C$ maps to V. Then $f^{-1}(V)$ contains C' but not p, and so $f^{-1}(V)$ is not closed. The converse follows from the continuity of f and the inclusions $X_d \subset X$.

Another consequence is the theorem of Bergman and Small [4]:

COROLLARY 5.15. *Let V be a constructible subset of $X_n = \operatorname{Spec}_n R$, and let $B = \bar{V} - V$ be its boundary. If $B_d = B \cap X_d$ is non-empty, then there are positive integers $d_1, \ldots, d_l, v_1, \ldots, v_l$ with $0 < d_i \leqslant m$ and $d_1 = d$, such that*

$$B_{d_i} \neq 0, \qquad \textit{and} \qquad n = \sum d_i v_i.$$

In fact, if $x \in B_d$ is an accessible boundary point, there is a curve $C \xrightarrow{f} X$ passing through $x = f(p)$, with a non-empty open subset C' mapping to V. Since $V \subset X_n$, the pi degree of C is n, and so the corollary follows from

Formula (2.8). In general, there may be no accessible point in B_d, even though the accessible points are dense in B. But if W denotes the constructible set of accessible points in B, then $B \subset \bar{W} = \bar{W}_1 \cup \cdots \cup \bar{W}_n$. So any $x \in B_d$ is in the boundary of $W_{d'}$ for some d'. By what has been shown, d' satisfies the conclusion of the corollary, since $W_{d'} \neq \varnothing$. By Noetherian induction on $\bar{V}$, we can assume the corollary holds for $V' = W_{d'}$. Thus n can be expressed as a linear combination involving d', and d' can be expressed similarly in terms of d, so by substitution we have an expression for n in terms of d.

Proposition 5.16. *Let $Y \xrightarrow{f} X$ be a central map, and let $C \xrightarrow{g} X$ be a curve in X. Assume that $g^{-1}(f(Y))$ is dense in C. Let $C = \operatorname{Spec} A$, where A is a D-order. There exists an injection $D \hookrightarrow D'$ of Dedekind domains, and a commutative diagram*

$$\begin{array}{ccc} C' & \xrightarrow{g'} & Y \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & X, \end{array}$$

where $C' = \operatorname{Spec} D' \otimes_D A$.

Proof. Say that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. We may replace X by C and Y by the pull-back $\operatorname{Spec}(A \otimes_R S) = C \times_X Y$. Then using Proposition 4.16 we may replace Y by a one-dimensional closed subscheme. At this point, we have a central homomorphism $A = R \xrightarrow{\varphi} S$, with $\dim S = 1$, and the fact that $f(Y)$ is dense in $C = Y$ translates into the injectivity of φ. Since A is a prime ring, we may replace S by one of its prime quotients.

Let $Z = Z(S)$. Then Z is prime, of dimension 1, and so some localization D' is a Dedekind domain. Replace S by the localization $D' \otimes_D S$. The D'-algebra $D' \otimes_D A$ is flat and hence a D'-order. Since S is central over A, it is a quotient of $D' \otimes_D A$. Since $D' \otimes_D A$ and S are prime rings of dimension 1, $D' \otimes_D A \approx S$.

6. The Curve Criterion for Integrality

Let $R \xrightarrow{\varphi} S$ be a homomorphism of commutative rings, and let $X \xleftarrow{f} Y$ be the associated map between their spectra. A criterion of Chevalley [5, p. 280, exc. 11.2] asserts that φ is integral if and only if f is a proper map of schemes. Moreover, the valuative criterion [5, p. 107, exc. 4.11] for properness can be stated in the following way:

Let C be a (commutative) curve, and let $C' = C - \{p\}$ for some point $p \in C$. Every commutative diagram of solid arrows

$$\begin{tikzcd} C' \arrow[r, "g'"] \arrow[d, phantom, "\cap"] & Y \arrow[d, "f"] \\ C \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \tag{6.1}$$

can be completed by a unique broken arrow.

In other words, the map g' can be extended to the point p if and only if fg' can. We propose to generalize this criterion to homomorphisms $R \to^{\varphi} S$ of pi rings. Obviously, the fact that a ring homomorphism will usually induce a multi-valued correspondence $X \leftarrow^{f} Y$ must be taken into account, and this complicates the geometric statement of the condition. It is more convenient to work with an equivalent algebraic formulation.

To state the curve criterion, we use the following notation: Let D be a Dedekind domain, $t \in D$ a non-zero element, and $D' = D[t^{-1}]$. Let A' be a D'-order. Given a homomorphism $S \to A'$, we will denote by $S[D]$ the subring of A' generated by the image of S and D.

DEFINITION 6.2. A homomorphism $R \to^{\varphi} S$ is called *proper* if it satisfies the following *curve criterion*: For every curve $S \to^{\lambda} A'$, where A' is as above, if t is invertible in $S[D]$ then it is invertible in $R[D]$.

Note that $t^{-1} \in S[D]$ if and only if $S[D] = A'$.

THEOREM 6.3. *Let $R \to^{\varphi} S$ be a ring homomorphism. The following conditions are equivalent*:

(a) *φ is integral.*

(b) *φ is proper.*

(c) *If $\bar{S}$ is a prime quotient of S of Krull dimension* 1, *every element α of its center $Z(\bar{S})$ is integral over R.*

Conditions (b) and (c) are useful because they reduce the verification of integrality to one-dimensional prime rings, which are relatively easy to control. Note that, in particular, (c) implies that φ is integral if for every prime one-dimensional quotient $\bar{S}$ of S the map $R \to^{\bar{\varphi}} \bar{S}$ is integral.

The only trivial implication among the conditions is (a) $\Rightarrow$ (c). However the equivalence of (b) and (c) is relatively elementary and is proved in this section. The more difficult part, the implication (b) $\Rightarrow$ (a), is proved in Section 8.

Even for prime rings of dimension 1, the theorem is non-trivial and interesting.

EXAMPLE 6.4. We return to Example 1.5 in which $R = k[x]$, $S = M_2(k[u])$. As we have seen, the correspondence f has a graph in the (x, u)-plane with the locus

$$x^2 - (\operatorname{tr} \alpha)\, x + (\det \alpha) = 0,$$

and the equivalence of (a) and (c) of Theorem 6.3 shows that φ is integral if and only if this equation defines u as an integral function of x. The integral equations for arbitrary elements of S are not obvious in case (1.6) even though φ is the composition of the homomorphisms $k[x] \to k[\alpha, u] \to M_2(k[u])$, each of which is obviously integral.

One has considerable freedom to manipulate the Dedekind domain D in the curve criterion. The following proposition is elementary, and we omit the proof.

PROPOSITION 6.5. (i) *To show that a homomorphism φ is proper, it suffices to verify the curve criterion in the case that t generates a prime ideal in D.*

(ii) *Let $S \to^{\lambda} A'$ be as in Definition 6.2, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the primes of D containing t. To verify the curve criterion for λ, one may choose injections of Dedekind domains $D \to E_i$ such that $\mathfrak{p}_i E_i < E_i$, and verify it for each i when D, A' are replaced by E_i, $E_i \otimes_D A'$ respectively.*

Remark 6.6. Suppose that t generates a prime ideal $\mathfrak{p}$ in D. Then $S[D] = A'$ if $t^{-1} \in S[D]$, and otherwise $S[D]$ is a D-order. This follows easily from Proposition 2.2.

For the sake of completeness, we include a proof of Chevalley's theorem:

PROPOSITION 6.7 (Chevalley). *Properness is equivalent with integrality for a homomorphism $R \to^{\varphi} S$ of commutative rings.*

Proof. Note that for commutative rings we have $A' = D'$ in the curve criterion, and so it reads as follows: Let $S \to^{\lambda} D'$ be a homomorphism. If t is not invertible in D, it is not invertible in $S[D]$.

Suppose S is integral over R. Then $S[D]$ will be integral, hence a finite module, over D. The condition then follows by the Nakayama lemma. Thus integrality implies that φ is proper.

Conversely, suppose that φ is proper. To show that a given element $\alpha \in S$ is integral, we may replace S by the subring $R[\alpha]$: the curve criterion carries over. So assume $S \approx R[u]/I$. We change coordinates to $u^{-1} = t$. Let

$I'' = IR[u, t]$, and let $I' \subset R[t]$ be the inverse image of I''. Thus I' is the set of polynomials $f(t)$ such that $u^N f(t) \in I$ if $N \gg 0$. We will use the curve criterion to prove

LEMMA 6.8. $I' + tR[t] = R[t]$.

Assume that the lemma has been proved. Then we can write $1 = f(t) + tg(t)$, with $f(t) \in I$. Hence for any N,

$$u^N = u^N f(t) + u^{N-1} g(t).$$

Since $u^N f(t) \in I$ if N is large, this gives an integral equation for α over R.

Proof of Lemma 6.8. Let $Y' = V(I')$ and $X_\infty = V(t)$ in $\operatorname{Spec} R[t]$. It follows from the definition of I' that $Y'' = Y' - (Y' \cap X_\infty)$ is dense in Y'. Suppose $Y' \cap X_\infty$ is not empty, and let $p \in Y' \cap X_\infty$. We may (Proposition 5.10) choose a curve $C = \operatorname{Spec} D$ in Y' passing through p, and with $t \neq 0$ at all but a finite number of its points. Denote the image of t in D by t too. Consider the diagram of ring homomorphisms

$$\begin{tikzcd} S \arrow[r, "\sigma"'] & R[u,t]/I'' \arrow[r, "\tau"'] & D' = D[t^{-1}] \\ R \arrow[u] \arrow[r] & R[t]/I' \arrow[u] \arrow[r] & D. \arrow[u] \end{tikzcd}$$

Since t vanishes at p, it is not invertible in D. But it has the inverse $\lambda(\alpha) = \tau(u)$ in $S[D]$. This contradicts the curve criterion for $\lambda = \tau\sigma$ and proves the lemma.

PROPOSITION 6.9. *The curve criterion* (*Definition* 6.2) *implies the following more general property*: *Let D be a semi-prime commutative ring of Krull dimension* 1, *$t \in D$ a non-zero-divisor, and $D' = D[t^{-1}]$. Let A' be a finite D'-algebra. Suppose a homomorphism $S \to^{\lambda} A'$ is given. If t is invertible in $S[D]$, then it is invertible in $R[D]$.*

Proof. We replace A' by $S[D']$, thereby reducing to the case that the homomorphism λ is central. Next, we may replace A' by A'/J', where J' is its (nilpotent) Jacobson radical. Thus we may assume λ central and A' semi-prime. Let Z' be the center of A', and let Z be the integral closure of D in Z'. Then $Z' = Z[t^{-1}]$, and Z is finite over D. It follows that t is invertible in $R[D]$ if it is invertible in $R[Z]$, hence that we may replace D by Z, i.e., assume that D' is the center of A'. Let D_1 be the integral closure of D. This is a finite product of Dedekind domains and copies of k, and is a finite D-algebra. We may replace D by D_1 and A' by $A'_1 = D_1 \otimes_D A'$. Again passing to the semi-prime ring A'/J', we are left with the case that A' is a product of orders and of matrix algebras over k. Taking each factor separately, the orders lead to the curve criterion, and for matrix algebras the conclusion of the proposition is trivial.

PROPOSITION 6.10. (i) *Let* $R \to^{\varphi} S$ *be a homomorphism, and let* $\mathfrak{a} \subset R$, $\mathfrak{b} \subset S$ *be ideals such that* $\varphi(\mathfrak{a}) \subset \mathfrak{b}$. *If* φ *is proper, so is the induced map* $R/\mathfrak{a} \to S/\mathfrak{b}$.

(ii) *Let* $R_1 \to^{\varphi_1} R_2 \to^{\varphi_2} R$ *be homomorphisms. If* φ_1 *and* φ_2 *are proper, so is* $\varphi_2\varphi_1$.

(iii) *Let* $R \to^{\varphi} S$ *be a proper homomorphism, and let* u *be a central variable. Then the induced homomorphism* $R[u] \to S[u]$ *is proper.*

Proof. Assertion (i) is trivial from the definition, and (ii) follows from the formulation of Proposition 6.9. To prove (iii), consider a test $S[u] \to A'$ of the curve criterion. We may assume that t generates a prime ideal in D (Proposition 6.5(i)). Suppose $t^{-1} \in S[u, D]$. The subring $S[D]$ of A' is a prime ring of Krull dimension 1 because $S[D'] = A'$. Its center Z lies between D and D'. Therefore $Z = D'$ or $Z = D$. In the first case, $t^{-1} \in S[D]$ and the curve criterion for φ shows that $t^{-1} \in R[D] \subset R[u, D]$. In the second case, $t^{-1} \in S[u, D]$ implies $u \notin D$; hence $D[u] = D'$, and $t^{-1} \in R[u, D]$.

We now proceed to the proof of the equivalence of (b) and (c) in Theorem 6.3.

LEMMA 6.11. *Properness implies integrality for a central homomorphism* $R \to^{\varphi} S$, *when* S *has Krull dimension* $\leqslant 1$.

Proof. Assume that φ is proper. We may suppose that S is semi-prime and that φ is injective, hence that R is also semi-prime. The center $Z(R)$ is a subring of $Z(S)$, and therefore has dimension $\leqslant 1$. So R has dimension $\leqslant 1$ too (because Krull dimension can be expressed in terms of transcendence degree). Since S is finite over $Z(S)$ (Proposition 2.1), it is enough to show that $Z(S)$ is finite over $Z(R)$. We do this by showing that the curve criterion holds for $Z(R) \to Z(S)$, and applying Proposition 6.7.

Consider a test map $Z(S) \to D'$, and suppose $t^{-1} \in Z(S)[D]$. Let $A' = D' \otimes_{Z(S)} S$. We apply Proposition 6.9 to the map $S \to A'$, to conclude that $t^{-1} \in R[D]$. Since R is finite over D (Proposition 2.1), $t^{-1} \in D$.

LEMMA 6.12. *Let* $R \to^{\varphi} S$ *be a central homomorphism.*

(i) *If* φ *is integral then* S *is a finite central* R*-module.*

(ii) *Assume* S *to be one-dimensional. If* S *is a finite central* R*-module, then* φ *is integral.*

(iii) *Assume that* R, S *are semi-prime and one-dimensional.*

Then φ *is integral if and only if* $Z(S)$ *is integral over* $Z(R)$.

Proof. Assertion (i) is elementary. Also (ii) is valid without the hypothesis that S be one-dimensional (Theorem 1.1). In our case, however, we can argue as follows. It is easily seen that S is integral over R if and only if S/J is, where J is the Jacobson radical. Thus we may assume that S is semi-prime. Replacing R by its image in S, we may also assume that R is semi-prime. Then by Proposition 2.1(i), R, and hence S, is a finite algebra over $Z(R)$, which implies that S is integral. Assertion (iii) follows again from Proposition 2.1.

LEMMA 6.13. *Conditions* (b) *and* (c) *of Theorem* 6.3 *are equivalent.*

Proof that (b) $\Rightarrow$ (c). We may replace S by a one-dimensional prime quotient $\bar{S}$, and then show that $Z(S) = Z$ is integral over R. Let $R' = R[Z]$ be the subring of S generated by R and Z. The homomorphism $R \to^{\psi} R'$ is central and R' has dimension 1. By Lemma 6.11, the map ψ is integral if the curve criterion holds for it. So suppose a test $R' \to^{\lambda} A'$ is given and denote the induced map $Z \to D'$ by τ. We may assume (Proposition 6.5) that t generates a prime ideal in D. Since Z is a one-dimensional domain, there are two possibilities: either τ factors through k: $Z \to k \to D'$, or else the field of fractions Fract(D') is a finite extension of Fract(Z).

In the first case, since R' is finite over Z, $\lambda(R')$ is a finite k-algebra, and $R'[D]$ is finite over D. Thus $t^{-1} \in R'[D]$ implies $t^{-1} \in D \subset R[D]$.

In the second case, consider the diagram

$$\begin{tikzcd} R \arrow[r, "\psi"] & R' \arrow[r] \arrow[d] & S \arrow[d] \\ & D' \otimes_Z R' \arrow[r, "c"] \arrow[d, "a"] & D' \otimes_Z S \arrow[d, "b"] \\ & A' & B', \end{tikzcd} \tag{6.14}$$

where B' is the unique prime one-dimensional quotient of $D' \otimes_Z S$. Note that since Z is the center of the prime ring S, B' will be an order over D' (Proposition 2.1(ii)), and the kernel of b will be D'-torsion (and hence finite over k). The kernel of c is also D'-torsion, and so since A' is torsion-free,

$$\ker(a) \supset \ker(bc). \tag{6.15}$$

We may apply the curve criterion for φ to the map $S \to B'$ of (6.14): If $t^{-1} \in S[D]_{B'}$ then $t^{-1} \in R[D]_{B'}$. (We are using a subscript here, to indicate the ring in which we work.) By (6.15), $R[D]_{A'}$ is a quotient of $R[D]_{B'}$, hence $t^{-1} \in R[D]_{A'}$. So we are done in that case. Suppose $t^{-1} \notin S[D]_{B'}$. Then since t generates a prime ideal in D, $S[D]_{B'}$ is a D-order, finite over D. It follows that the rings generated by D and R' in B', $D' \otimes_Z S$, $D' \otimes_Z R'$ are finite over D, hence that $R'[D]_{A'}$ is finite over D. Then $t^{-1} \in R'[D]_{A'}$ implies $t^{-1} \in D \subset R[D]$, as required.

Proof that (c) $\Rightarrow$ (b). We assume (c), and consider a test of the curve criterion $S \to^{\lambda} A'$ in which (Proposition 6.5) t generates a prime ideal of D. Let $\bar{S}$ denote the image of S; it is a prime ring of dimension $\leqslant 1$. The curve criterion is trivial in the case $\dim \bar{S} = 0$, so assume $\dim \bar{S} = 1$. Replace S by $\bar{S}$, and let $Z = Z(S)$. Thus every element of Z is integral over R, by (c). Since S is finite over Z, $S[D]$ is finite over $Z[D]$.

Suppose $t^{-1} \in S[D]$. Then t^{-1} is integral over $Z[D]$, and since $Z[D]$ is a Dedekind domain, $t^{-1} \in Z[D]$. Let a bar denote reduction modulo the Jacobson radical. The inclusions

$$\begin{array}{ccc} R[Z] & \subset & R[Z, D] \\ \cup & & \cup \\ R & \subset & R[D] \end{array}$$

remain inclusions when bars are added, because they are central. By Lemma 6.11, $\overline{R[Z]}$ is a finite module over $\bar{R}$, hence $\overline{R[Z, D]}$ is a finite module over $\overline{R[D]}$. Let $\bar{t}$ be the residue of t. Then $\bar{t}^{-1} \in \overline{R[Z, D]}$; hence it is integral over $\overline{R[D]}$. Since $\bar{t}$ is central in $\overline{R[D]}$, $\bar{t}^{-1} \in \overline{R[D]}$. Since the Jacobson radical of $R[D]$ is nilpotent, $t^{-1} \in R[D]$. This completes the proof.

7. Geometric Characterization of Properness

The following proposition shows that Definition (6.2) of proper homomorphism is equivalent to a geometric condition which mimics the one used in algebraic geometry [5, p. 100].

Proposition 7.1. (i) *Let* $R \to^{\varphi} S$ *be a proper homomorphism. The induced correspondence* $\operatorname{Spec} S \to^{f} \operatorname{Spec} R$ *is closed, i.e., carries closed sets to closed sets, and has finite fibers. Moreover, if* $\{u_1, \ldots, u_m\}$ *are central variables, then the correspondence* $\operatorname{Spec} S[u] \to \operatorname{Spec} R[u]$ *is closed and has finite fibers.*

(ii) *Conversely, if* u *is a single variable, and if the correspondence* $\operatorname{Spec} S[u] \to \operatorname{Spec} R[u]$ *is closed, then* φ *is proper.*

Note. In (ii), it would not suffice to assume $\operatorname{Spec} S \to \operatorname{Spec} R$ closed. For example, any homomorphism of Dedekind domains which induces a surjective map of spectra has that property.

COROLLARY 7.2. *Assume that $R \to^{\varphi} S$ is injective and proper. Then the induced correspondence f is surjective.*

In fact, since φ is injective the image of f is dense (Proposition 4.6(vi)). By Proposition 7.1 it is closed, and hence is all of Spec R.

Proof of Proposition 7.1(i). Assume that $R \to^{\varphi} S$ is proper. By Proposition 6.10(iii) $R[u] \to S[u]$ is proper. So it suffices to treat the correspondence induced by φ itself. We denote it as usual by $\operatorname{Spec} S = Y \to^{f} X = \operatorname{Spec} R$. To show f closed, let $V \subset Y$ be a closed set. If $V = V(I)$, where I is semi-prime, we may replace S by S/I (Proposition 6.10(i)), and are therefore reduced to the case that $V = Y$ and that S is semi-prime. We may also assume φ to be injective (Proposition 6.10(i)). Thus we are reduced to showing that f is surjective if φ is injective and proper, and if S is semiprime (i.e., to verifying the corollary). With these assumptions the image of f is constructible (4.1) and dense (Proposition 4.6(vi)).

Suppose $f(Y) < X$. Then (Theorem 5.13) there is an accessible point p in the boundary of $f(Y)$, and we may choose a curve C in X passing through p and with all but a finite number of points in $f(Y)$. Let the curve be defined with the usual notation by $R \to^{\lambda} A$, let $\mathfrak{p} = \ker \lambda$ and $R_1 = R/\mathfrak{p}$. Then R_1 is one-dimensional and a dense subset of $\operatorname{Spec} R_1 \subset X$ is in $f(Y)$. Therefore (Proposition 4.16) there is a prime ideal $\mathfrak{q} \subset S$ such that $\bar{S} = S/\mathfrak{q}$ has dimension 1 and $\mathfrak{p} \supset \varphi^{-1}(\mathfrak{q})$. By Proposition 6.10(i), we may replace S by $\bar{S}$ and R by $R/\varphi^{-1}(\mathfrak{q})$, thus reducing to the case that φ is injective and S is a prime ring of dimension 1. We still have to show that f is surjective.

By Lemma 6.12 the center Z of S is integral over R. Since Z is finitely generated (Proposition 2.1), $R[Z]$ is a finite central R-module, and the Nakayama lemma [1, (5.2)] shows that the map $\operatorname{Spec} R[Z] \to X$ is surjective. Since $R[Z]$ and S are finite Z-modules, the correspondence $Y \to \operatorname{Spec} R[Z]$ is surjective (Proposition 4.15). Finally, the composition of surjective correspondences is surjective. This shows that f is surjective in the case under consideration, and therefore closed in general.

It remains to show that f has finite fibers. Suppose some $f^{-1}(x)$ is not finite. Then since the fiber is constructible (Corollary 4.14), it contains an irreducible closed subset of dimension 1. Let $\mathfrak{q} \subset S$ be the corresponding prime ideal. We again replace S by $S/\mathfrak{q}$ and R by $R/\varphi^{-1}(\mathfrak{q})$. The considerations of the previous paragraph show that the fibers in this case are finite. This is a contradiction which completes the proof of Proposition 7.1(i).

Proof of Proposition 7.1(ii). Let A' be an order over $D' = D[1/t]$, and let $S \to^{\lambda} A'$ be a test curve. Since $\dim D \leqslant 1$, we can find a $u \in D$, so that D is integral over $k[u]$. Let $\bar{S}[u] \subseteq \bar{S}[D]$ be the subrings of A' generated by $\lambda(S)$ and u (resp. $\lambda(S)$ and D). Let $R[u] \subseteq R[D]$ be defined similarly. Take $t \in D$, and assume $t^{-1} \in \bar{S}[D]$. We may assume tD is maximal, $tD \cap k[u] = (u)$, and tD is the only maximal ideal of D over (u). Thus $t^n = ud$, for some $d \in D$. So if $t^{-1} \in \bar{S}[D]$ then $u^{-1} \in \bar{S}[d]$. Now the image of $\operatorname{Spec} \bar{S}[u]$ is dense in $\operatorname{Spec} \bar{R}[u]$, and hence equal to $\operatorname{Spec} \bar{R}[u]$ since it is closed. Thus u is non-zero at every point of $\operatorname{Spec} \bar{R}[u]$, so $u^{-1} \in \bar{R}[u]$. Hence $t^{-1} \in \bar{R}[D]$, since $u \in tD$. We have thus verified the curve criterion.

PROPOSITION 7.3. *Let $F \to^{\psi} R \to^{\varphi} S$ be homomorphisms. Assume that φ is injective and $\varphi\psi$ is proper. Then ψ is proper.*

Proof. Let $R_1 = R/\mathfrak{p}$ be a one-dimensional quotient of R. We have to show that $Z(R_1)$ is integral over F (Theorem 6.3). As in the previous proof, we apply Proposition 4.16 to find a prime ideal $\mathfrak{q} \subset S$ such that $\mathfrak{p} \supset \varphi^{-1}(\mathfrak{q})$ and $\bar{S} = S/\mathfrak{q}$ is one-dimensional. We may replace S by $\bar{S}$, R by $R/\varphi^{-1}(\mathfrak{q})$, and F by $F/(\varphi\psi)^{-1}(\mathfrak{q})$.

By assumption, $D = Z(S)$ is integral over F, hence over R. So the subring $R[D]$ of S is a finite central R-module, and a finite D-module as well. Consider the finite central R_1-module $R[D]/\mathfrak{p}R[D] = R_2$. Apply Proposition 1.10 to the map $R \to R[D]$ to find a prime ideal of $R[D]$ lying over P. Thus R_1 maps injectively to R_2:

$$F \longrightarrow R_1 \hookrightarrow R_2. \tag{7.4}$$

Moreover, R_2 is a finite D-module. Let $E = Z(R_2)$. Then E is a finite D-module. Thus each of the central homomorphisms

$$F \to F[D] \to F[E]$$

makes the range into a finite module over the domain. Therefore $F[E]$ is a finite central F-module too. This implies (Lemma 6.12(ii)) that E is integral over F. Since $Z(R_1) \subset E$, that ring is also integral over F, as required.

PROPOSITION 7.5. *Let $R \to^{\varphi} S$ be a proper homomorphism. Then for every n, the homomorphism $k[T_n(R)] \to^{\tau} k[T_n(S)]$ is integral.*

Proof. To show that τ is integral, we use Chevalley's criterion (6.1). Let us write $T = T_n(S)$, and let $k[T] \to^{\lambda} D'$ be a test of the curve criterion for τ. This homomorphism corresponds to a family of equivalence classes of representations of S, which we can realize using Propositions 3.24, 5.7 by a collection of curves $S \to^{\prod \sigma_v} \prod A'_v$, the A'_v being orders over D'. This may require a change of parameter in a finite field extension of $K = \operatorname{Fract}(D')$, but that is permissible by Proposition 6.5(ii). To apply Proposition 3.24, we let A' be the subring of $\bigoplus_v M_{t_v}(K)$ generated by $R[T]$ and D', and let A'_v be the projection to $M_{t_v}(K)$. Proposition 5.7 implies that A', and hence A'_v, is finite over D'. Therefore A'_v is a D-order.

We may also assume D' has the form $D[t^{-1}]$, and that t generates a prime ideal in D (Proposition 6.5(i)). Suppose t^{-1} is in the subring $k[T, D]$ of D'. It follows from Proposition 5.7 that at least one of the curves σ_ν fails to extend over Spec D. For that ν, the subring $S[D]_\nu$ generated by S and D in A'_ν is not a D-order, hence (Remark 6.6) $t^{-1} \in S[D]_\nu$. Since φ is proper, $t^{-1} \in R[D]_\nu$, and so $R[D]_\nu$ is not a finite D-module. By Proposition 5.7(ii), there is an $\alpha \in R$ such that at least one eigenvalue of $\sigma_\nu(\alpha)$ has a pole at $t = 0$. Let $\bar{\alpha}$ denote the $n \times n$ matrix $[\prod \sigma_\nu](\alpha)$. Then for all but a finite number of $c \in k$, the eigenvalues of $\bar{\alpha} + c$ are non-zero at $t = 0$, but at least one has a pole. Therefore $\det(\bar{\alpha} + c) = \lambda(\det(\alpha + c))$ has a pole too, and so $k[T_n(R)][D] \neq D$. Since t generates a prime ideal in D, it follows that $t^{-1} \in k[T_n(R)][D]$, as required.

8. Proof That a Proper Homomorphism Is Integral

The part of Theorem 6.3 which remains to be proved is that properness implies integrality. We now prove the following stronger assertion:

THEOREM 8.1. *Let $R \to^{\varphi} S$ be a proper homomorphism. Let $s_1, ..., s_k$ be elements of S, and denote the subring they generate over R by $S_0 = R\{s_1, ..., s_k\}$. There is an integer m with this property: Every $\alpha \in S_0$ can be written as a polynomial in $\{s_1, ..., s_k\}$ with coefficients in R, and of degree $< m$.*

Here the word polynomial is used in the generalized sense as in Section 1, with coefficients interspersed among the variables.

Note that the integer m in Theorem 8.1 is allowed to depend on $\{s_1, ..., s_k\}$. We do not know if there exists a uniform bound.

COROLLARY 8.2. *Integrality implies the stronger conclusion of Theorem* 8.1.

This follows from Theorem 8.1 and the implication (a) $\Rightarrow$ (b) (already proved) of Theorem 6.3.

Proof of Theorem 8.1. By *degree* of an element α, we mean the smallest degree of a polynomial expression in $\{s_i\}$ for α. To prove the theorem, it is enough to find m so that every evaluation on $\{s_1, ..., s_k\}$ of a monomial of degree m,

$$\alpha = r_0 s_{i_1} r_1 s_{i_2} s_2 \cdots s_{i_m} r_m, \qquad r_\nu \in R,$$

can be expressed as a sum of elements of lower degree.

Proposition 7.3 shows that we may assume $S_0 = S$.

LEMMA 8.3. *To prove Theorem* 8.1 *for rings S of Krull dimension $\leqslant d$, it suffices to do so when in addition S is a prime ring.*

Proof. Assume that we have ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset S$ with $\mathfrak{a}_1 \mathfrak{a}_2 = 0$, and that the theorem is true for $S/\mathfrak{a}_i$ with the integer $m = l$ (cf. Proposition 6.10(i)). Let $\alpha \in S$ be an evaluation of a monomial (Corollary 8.2) of degree $2l$. We may write it as a product $\alpha = \alpha_1 \alpha_2$ of monomials of degree l. Then since l is a bound for $S/\mathfrak{a}_i$, there are elements $\beta_1, \beta_2 \in S$ of degree $< l$, so that $(\alpha_i - \beta_i) \in \mathfrak{a}_i$. It follows that $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) = 0$, and hence that

$$\alpha = \alpha_1 \alpha_2 = \alpha_1 \beta_2 + \beta_1 \alpha_2 - \beta_1 \beta_2.$$

The right side of this equation, and hence α, has degree less than $2l = m$. Thus Theorem 8.1 holds for S, with $m = 2l$.

Since the Jacobson radical J of S is nilpotent [11; 13, Theorem 2], induction shows that the theorem is true for S if it is true for S/J. This reduces to the case that S is semi-prime. A similar argument reduces to the case that S is a prime ring.

Assume now that S is prime and of pi degree n. Let $h(x_1, ..., x_\mu) = h(x)$ be a central polynomial which is a conductor for matrices of rank n. We recall (3.4) that this is true if $h(x)$ is a sufficiently large power of any central polynomial, and there is an $n \times n$ matrix identity of the form

$$h(x) \det y = g(x, y), \tag{8.4}$$

for some polynomial g in $\{x_1, ..., x_\mu; y\}$. Let c be a non-zero evaluation of h in S. Then since $S = R\{s_1, ..., s_k\}$,

$$c = h(x(s)),$$

for some polynomials $x_\nu(w_1, ..., w_k)$, in variables $\{w_i\}$, and with coefficients in R. Choose an integer p_0 so that

$$p_0 > \text{degree in } \{w_i\} \text{ of } h(x(w)) \text{ and of } g(x(w), y). \tag{8.5}$$

We now wish to work with the homomorphism $R[T_n(R)] \to S[T_n(S)]$. Since S is of pi degree n and prime, we can consider the canonical prime quotient $S[\bar{T}]$ of $S[T_n(S)]$. It is convenient to work in this ring, so we replace rings by their images there, denoting such an image by a star: $R[T_n(R)]^*$, etc. Of course, the map $S \to S[\bar{T}]$ is injective, so $S \approx S^*$.

By Proposition 7.5, $k[T_n(S)]$ is integral over $k[T_n(R)]$, and by Proposition 3.10, $S[T_n(S)]$ is a finite module over $k[T_n(S)]$. Therefore $S[T_n(S)]$ is a finite module over $k[T_n(S)]$, and so is the subring $S[T_n(R)]^*$ generated by S^* and $k[T_n(R)]^*$. Every element of this last ring is of the form $\alpha = \sum z_i \sigma_i$, where $z_i \in k[T_n(R)]^*$ and $\sigma_i \in S$. Therefore $S[T_n(R)]^*$ is generated as module by a finite set $\{\sigma_\nu\}$ of elements of S. Let q be a bound for their degrees in $\{s_i\}$ over R.

Let $z \in k[T_n(R)]^*$. Then (Proposition 3.3) z can be represented as a sum of determinants

$$z = \sum_\nu \det \alpha_\nu, \qquad \alpha_\nu \in R.$$

Applying (8.4), we find that

$$h(x)^j z = h(x)^{j-1} \sum_\nu g(x, \alpha_\nu)$$

in the ring $R[T_n(R)]^*\{x\}$, and

$$c^j z = c^{j-1} \sum_\nu g(x(s), \alpha_\nu)$$

in $S[T_n(R)]^*$. Since $\alpha_\nu \in R$, the right side of this equation is an element of S whose degree in $\{s_i\}$ is bounded by $(p_0 - 1)j$:

$$\deg c^j z \leqslant (p_0 - 1)j. \tag{8.6}$$

We apply induction on the Krull dimension and Lemma 8.3, to reduce to the case that the theorem is true for the ring S/cS. Thus there is an integer p with the property that every $\alpha \in S$ is congruent (modulo cS) to an element of degree $< p$. We may assume $p \geqslant p_0$. Then we claim that the theorem is true for S, with the integer $m = pq$.

Let α be a monomial of degree m, and write it as a product $\alpha = \alpha_1 \cdots \alpha_q$ of monomials of degree p. Then there are elements $\beta_i \in S$ with $\deg \beta_i < p$, so that $\alpha_i - \beta_i \equiv 0$ (modulo cS). Hence

$$(\alpha_1 - \beta_1) \cdots (\alpha_q - \beta_q) = c^q \gamma, \tag{8.7}$$

for some $\gamma \in S$. We pass to the ring $S^* \subset S[T_n(R)]^*$, and write γ in terms of the generators $\{\sigma_\nu\}$:

$$\gamma = \sum z_\nu \sigma_\nu, \qquad z_\nu \in k[T_n(R)].$$

Then by (8.6),

$$\deg c^q \gamma \leqslant \max(\deg c^q z_\nu \sigma_\nu) < (p-1)\, q + q = m. \tag{8.8}$$

This, together with formula (8.7), shows that $\deg \alpha < m$ as required, and completes the proofs of Theorems 8.1 and 6.3.

9. The Case of a Geometric Homomorphism

In this section we show that for certain homomorphisms, checking integrality of a finite number of elements and the coefficients of their characteristic polynomials is sufficient. We first given an example.

EXAMPLE 9.1. All the small monomials in the generators of R and S are integral over R, as are their traces, but S is not integral over R. Let

$$R = \begin{pmatrix} k[u,v] & 0 \\ 0 & k[u,uv^{10}] \end{pmatrix} \subseteq S = R\left\{\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}\right\}.$$

Then small ($\deg \leqslant 10$) monomials are integral, but

$$\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v^{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & uv^{11} \end{pmatrix}$$

is not integral over R.

A ring homomorphism $R \to^{\varphi} S$ is called *geometric* if for each maximal ideal $\mathfrak{m}$ of S, the map $R/\varphi^{-1}(\mathfrak{m}) \to S/\mathfrak{m}$ is bijective, or equivalently, if for each prime ideal $\mathfrak{p}$ of S, $\varphi^{-1}(\mathfrak{p})$ is prime and of the same pi degree [3, (1.1), (1.3)]. Certainly central homomorphisms are geometric. Also we have

LEMMA 9.2. *If R is prime, and S is a subring of the ring of quotients of R, generated by R together with the inverses of certain elements of R, then $R \to S$ is geometric.*

Proof. Let $\mathfrak{p}$ be a prime ideal of S, and let L be the quotient field of the center of $S/\mathfrak{p}$. Let $\mathfrak{q} = \mathfrak{p} \cap R$. Then we have a diagram of inclusions

$$\begin{array}{ccc} R/\mathfrak{q} & \subset & S/\mathfrak{p} \\ \cap & & \cap \\ (R/\mathfrak{q})[L] & \subset & S/\mathfrak{p}[L]. \end{array} \tag{9.2}$$

If $\alpha \in R$ is an element such that $\alpha^{-1} \in S$, then its image is invertible, and hence regular, in $S/\mathfrak{p}[L]$. Therefore its image is regular in $(R/\mathfrak{q})[L]$, and so since that ring is a finite L-algebra, it is invertible there. Also, $S/\mathfrak{p}[L]$ is generated over $(R/\mathfrak{q})[L]$ by such elements α. Thus $(R/\mathfrak{q})[L] = S/\mathfrak{p}[L]$. Therefore $(R/\mathfrak{q})[L]$ is a prime ring, and it is a central extension of $R/\mathfrak{q}$. This shows that $\mathfrak{q}$ is prime, and that pi $\deg(R/\mathfrak{q}) =$ pi $\deg S/\mathfrak{p}$. (cf. [6] in this connection.)

THEOREM 9.3. *Let $R \to^{\varphi} S$ be a geometric homomorphism, and let* $n = \text{pi deg } S$. *The following are equivalent*:

(i) $R \to S$ *is integral*;

(ii) $R[T_m(R)] \to S[T_m(S)]$ *is integral for* $m \leqslant n$;

(iii) $SS_m(S) \to SS_m(R)$ *is proper for* $m \leqslant n$.

Proof. (ii) and (iii) are always equivalent, by Theorems 3.20 and 6.3. Also (i) $\Rightarrow$ (ii) always holds, by Proposition 7.5. It remains to show (ii) $\Rightarrow$ (i). It suffices by Theorem 6.3 to show $R \to \bar{S}$ is integral for every one-dimensional prime quotient $\bar{S}$ of S. If m is the pi degree of $\bar{S}$, then the image $\bar{R}$ of R in $\bar{S}$ is also prime and of pi degree m, since $R \to S$ is geometric. Thus there are maps

$$\begin{tikzcd}
R[T_m(R)] \arrow[r, "\psi"] \arrow[d] & S[T_m(S)] \arrow[d] \\
\bar{R}[\bar{T}] \arrow[r, "\bar{\psi}"] & \bar{S}[\bar{T}] \\
\bar{R} \arrow[r] \arrow[u, hook, "\varepsilon"] & \bar{S}. \arrow[u, hook]
\end{tikzcd}$$

By assumption, ψ is integral, and $\bar{\psi}$ is a quotient of ψ, hence integral by Proposition 1.9(ii). Moreover, $\bar{R}$ is prime and of dimension 1. The central integral closure of $\bar{R}$ is an order, and hence contains the trace ring $\bar{R}[\bar{T}]$. Thus ε is integral. It follows that $\bar{S}$ is integral over $\bar{R}$, as required.

REFERENCES

1. M. ARTIN, On Azumaya algebras and finite dimensional representations of rings, *J. Algebra* **11** (1969), 532–563.
2. M. ARTIN, Specialization of representations of rings, *in* "Proceedings, International Symposium on Algebraic Geometry, Kyoto, 1977," pp. 237–247, Kinokuniya Book Store, Tokyo 1978.
3. M. ARTIN AND W. SCHELTER, A version of Zariski's Main Theorem for polynomial identity rings, *Amer. J. Math.* **101** (1979), 301–330.
4. G. BERGMAN AND L. W. SMALL, Pi degrees and prime ideals, *J. Algebra* **33** (1975). 435–462.
5. R. HARTSHORNE, "Algebraic Geometry," Springer–Verlag, New York, 1977.
6. R. INDIK, A pi algebra which is not pi when an inverse is adjoined, *Proc. Amer. Math. Soc.* **76** (1979), 11–13.
7. N. JACOBSON, "Pi-Algebras," Lecture Notes in Mathematics No. 441, Springer–Verlag, Berlin 1975.
8. R. PARÉ AND W. SCHELTER, Finite extensions are integral, *J. Algebra* **53** (1978), 477–479.

9. C. Procesi, "Rings with Polynomial Identities," Dekker, New York, 1973.
10. C. Procesi, Finite dimensional representations of algebras, *Israel J. Math.* **19** (1975), 169–182.
11. Y. Razmyslov, Jacobson radical in pi algebras, *Algebra and Logic* **13** (1974), 192–204 [transl.].
12. W. Schelter, Integral extensions of rings satisfying a polynomial identity, *J. Algebra* **40** (1976), 245–254; errata, **44** (1979), 576.
13. W. Schelter, Noncommutative affine pi rings are catenary, *J. Algebra* **51** (1978), 12–17.
14. A. Shirshov, On rings with identity relations, *Mat. Sb.* **43** (1957), 277–283.
15. C. C. Tsen, Divisionsalgebren über Funktionenkörper, *Göttinger Nachr.* (1933), 335–339.